Discrete Toeplitz Determinants and their Applications

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2014

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ACKNOWLEDGEMENTS

I would like to thank my PhD advisor Professor Jinho Baik. It has been my honor to be his first PhD student. I feel very lucky to have such a great advisor in my PhD life. He has supported me for many semesters and also the summers even before I became a PhD candidate; He has spent countless hours these years introducing me into the area of random matrix theory and teaching me how to do research; He has been always available when I need help; He always has great patience to listen to my ideas and to give corresponding suggestions and ideas. I am also thankful for him to provide a perfect model of a mathematician and a professor. I will always remember all those moments we worked together.

I am grateful to Professor Peter Miller. He funded me for Winter 2011 semester and we met many times during that semester. Our conversations broadened my knowledge in the area. He is one of the readers of my dissertation and his suggestions are very useful. I would also thank Professors Anthony Bloch, Joseph Conlon and Raj Rao Nadakuditi for serving on my committee.

My time at Ann Arbor was enjoyable in large part due to many friends here. I would like to thank Linquan Ma, Sijun Liu, Yilun Wu, Jingchen Wu, Yi Su, Xin Zhou and other friends for learning and playing together. I am also indebted to Jiarui Fei, for his hospitality when I first arrived at Ann Arbor and his great help in my first year here.

Finally, I would like to thank my family for all their love and encouragement. For

my parents in China who supported me in all my pursuits. For my sister who took care of my parents in these years. For my parents in-law who come here twice to help us during the busy days. For my adorable and faithful wife Yu Zhan and my lovely children Kaylee and Brayden, who have been sources of my strength and courage to overcome all the hardships throughout my study.

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CHAPTER I

Introduction

In this dissertation, we consider the asymptotics of discrete Toeplitz determinants. We first show how one can convert this question to the asymptotics of *continuous* orthogonal polynomials by using a simple identity. We then apply this method to the width of nonintersecting processes of several different types. The asymptotic results on width can be naturally interpreted as an identity between the Airy process and the Tracy-Widom distribution from random matrix theory. We also prove several variations of this interesting identity.

Some parts of this thesis had already been published. Some portions of Chapters II, III and V were published in [14, 15]. The contents of Chapter IV and some parts of Chapter V are going to be included in [59, 30].

1.1 Discrete Toeplitz Determinant

Let $f(z)$ be an integrable function on the unit circle Σ and let $f_k := \int_\Sigma z^{-k} f(z) \frac{dz}{2\pi i z}$ be the Fourier coefficient of f, $k \in \mathbb{Z}$. The n-th *Toeplitz determinant* with *symbol* $f(z)$ is defined to be

$$(1.1) \qquad \mathcal{T}_n(f) = \det \left(f_{j-k} \right)_{j,k=0}^{n-1}.$$

Toeplitz determinants appear in a variety of problems in functional analysis, ran-

dom matrices, and many other areas in mathematics and physics. One of the most interesting questions is the asymptotic behavior of $\mathcal{T}_n(f)$ as $n \to \infty$. The first asymptotic result for Toeplitz determinants was obtained by Szegő in 1915 [71]. He proved that if f is a continuous positive function on Σ, then

$$(1.2) \qquad \lim_{n\to\infty} \frac{1}{n} \log \mathcal{T}_n(f) = \int_{\Sigma} \log(f(z)) \frac{dz}{2\pi i z}.$$

A few years after the Szegő's paper appeared, the correction term to (1.2) for a certain specific function f became an important question. This question was raised in the context of the two dimensional Ising model [61]. Szegő improved his previous argument and obtained a refined asymptotic result in [73], which is now called the *Szegő's strong limit theorem*: if f is positive and the derivative of f is Hölder continuous of order $\alpha > 0$, then

$$(1.3) \qquad \lim_{n\to\infty} \frac{\mathcal{T}_n(f)}{e^{n(\log f)_0}} = \exp\left(\sum_{k=1}^{\infty} k \, |(\log f)_k|^2 \right),$$

where $(\log f)_k$ denotes the k-th Fourier coefficient of $\log f$.

This result was generalized for a much larger class of functions over the following many years. For example, if $V(z) = \log(f(z))$ is a complex-valued function with Fourier coefficients V_k satisfying

$$(1.4) \qquad \sum_{k \in \mathbb{Z}} |k||V_k|^2 < \infty,$$

then [49]

$$(1.5) \qquad \lim_{n\to\infty} \frac{\mathcal{T}_n(e^V)}{e^{nV_0}} = e^{\sum_1^\infty k V_k V_{-k}}.$$

Another direction of generalizing (1.3) has been to consider the case when f contains singularities. Fisher and Hartwig [40] introduced the following class of symbols:

$$(1.6) \qquad f(z) = e^{V(z)} z^{\sum_{j=0}^{m} \beta_j} \prod_{j=0}^{m} |z - z_j|^{2\alpha_j} g_{z_j, \beta_j}(z) z_j^{-\beta_j}$$

where

$$(1.7) \qquad z_j = e^{i\theta_j}, j = 0, \cdots, m, \qquad 0 = \theta_0 < \cdots < \theta_m < 2\pi,$$

$$(1.8) \qquad g_{z_j,\beta_j}(z) = \begin{cases} e^{i\pi\beta_j}, 0 \le \arg z < \theta_j, \\ \\ e^{-i\pi\beta_j}, \theta_j \le \arg z < 2\pi, \end{cases}$$

$$(1.9) \qquad \Re\alpha_j > -\frac{1}{2}, \qquad j = 0, \cdots, m, \qquad \beta = (\beta_1, \cdots, \beta_m) \in \mathbb{C}^m,$$

and $V(z)$ is sufficiently smooth on Σ. They conjectured that for these symbols

$$(1.10) \qquad \mathcal{T}_n(f) = En^{\sum_{j=0}^m (\alpha_j^2 - \beta_j^2)} e^{nV_0} (1 + o(1))$$

as $n \to \infty$, where $E = E(e^V, \alpha_0, \cdots, \alpha_m, \beta_0, \cdots, \beta_m, \theta_0, \cdots, \theta_m)$ is an explicit function independent of n.

This conjecture was proved by Widom [79] for the case when $\beta_0 = \cdots = \beta_m = 0$. Widom's result was then improved by Basor [18] for the case when $\Re\beta_j = 0, j = 0, \cdots, m$, and Böttcher, Silbermann [24] for the case when $|\Re\alpha_j| < \frac{1}{2}, |\Re\beta_j| < \frac{1}{2}, j = 0, \cdots, m$. Finally Ehrhardt [38] proved the full conjecture under the following two additional conditions $|||\beta||| < 1$ and $\alpha_j \pm \beta_j \ne -1, -2, \cdots$ for all j, which were known to be necessary beforehand. Here

$$(1.11) \qquad |||\beta||| = \max_{j,k} |\Re\beta_j - \Re\beta_k|.$$

If the condition $|||\beta||| < 1$ is not satisfied, $\mathcal{T}_n(f)$ does not necessarily satisfy the asymptotics (1.10). For general β, the asymptotics of $\mathcal{T}_n(f)$ was conjectured to be

$$(1.12) \qquad \mathcal{T}_n(f) = \sum_{\hat\beta} R_n(f(\hat\beta))(1 + o(1))$$

by Basor and Tracy [19], where $R_n(f(\hat{\beta}))$ is the right hand side of (1.10) after replacing β by $\hat{\beta}$, the sum is taken for all $\hat{\beta}$ which is obtained by taking fintely many operations $(a, b) \rightarrow (a-1, b+1)$ for any two coordinates in β such that $|||\hat{\beta}||| \leq 1$, and $\alpha_j \pm \hat{\beta}_j \neq -1, -2, \cdots$ for all j. This conjecture was proved recently by Deift, Its and Krasovsky [34].

The function $f(z)$ sometimes contains a parameter, say t, and it is interesting to consider the double scaling limit of $T_n(f)$ as n and t both tend to infinity. This also has been studied for various function f. For example, in [9] the authors considered the distribution of the longest subsequence of a random permutation, which can be expressed in terms of a Toeplitz determinant with weight $f(z) = e^{t(z+z^{-1})}$. It turns out that the double scaling limit of this Toeplitz determinant multiplied by e^{-t^2} when the two parameters satisfy $n = 2t + xt^{1/3}$ is equal to the *GUE Tracy-Widom distribution* $F_{GUE}(x)$. F_{GUE} is a distribution which appears in random matrix theory, see Section 1.2 for more details.

In the cases above, the symbol f was assumed to be continuous. One of the goals of this dissertation is to study the asymptotic behavior of the Toeplitz determinant when its symbol is discrete. Let $\mathcal{D}$ be a discrete set on $\mathbb{C}$, and let f be a function on $\mathcal{D}$. The discrete Toeplitz determinant with measure $\sum_{z \in \mathcal{D}} f(z)$ is defined as

$$(1.13) \qquad \mathrm{T}_n(f, \mathcal{D}) := \det \left(\sum_{z \in \mathcal{D}} z^{-j+k} f(z) \right)_{j,k=0}^{n-1}.$$

Of course, the determinant is zero if $n \leq |\mathcal{D}|$. We assume that $|\mathcal{D}| \rightarrow \infty$ as $n \rightarrow \infty$.

The discrete Toeplitz determinants arise in various models. Some examples include the width of non-intersecting processes [14], the maximal crossing and nesting of random matchings [25, 11], the maximal height of non-intersecting excursions [57, 69, 44, 58], periodic totally asymmetric simple exclusion process [13], etc.

Discrete Toeplitz determinants contain two natural parameters, the size of the matrix and the cardinality of $\mathcal{D}$. The function f may also contain additional parameter, say t. It is sometimes interesting to consider the limit of (1.13) when all these parameters go to infinity. In Chapters II, III, and IV we develop a method to evaluate the limit of discrete Toeplitz determinants and apply it to the model of nonintersecting processes.

1.2 Random Matrices, the Airy Process and Nonintersecting Processes

Since the work of Wigner on the spectra of heavy atoms in physics in the 1950's, random matrix theory has evolved rapidly and became a prolific theory which has various applications in many areas including number theory, combinatorics, probability, statistical physics, statistics, and electrical engineering [60, 3, 43].

One of the most well-known random matrix ensembles is the Gaussian Unitary Ensemble (GUE). $\mathrm{GUE}(n)$ is described by the Gaussian measure

$$(1.14) \qquad \frac{1}{Z_n} e^{-n \operatorname{tr} H^2} dH$$

on the space of $n \times n$ Hermitian matrices $H = (H_{ij})_{i,j=1}^n$, where dH is the Lebesgue measure and Z_n is the normalization constant. This measure is invariant under unitary conjugations. Its induced joint probability density for the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ is given by

$$(1.15) \qquad \frac{1}{Z_n'} \prod_{k=1}^n e^{-n\lambda_k^2} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^2, \qquad (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n,$$

where Z_n' is the different normalization constant.

In the celebrated work [75], Tracy and Widom showed that the largest eigenvalue of $\mathrm{GUE}(n)$, after rescaling, converges to a limiting distribution which is now called

the Tracy-Widom distribution F_{GUE}:

$$(1.16) \qquad \lim_{n \to \infty} \mathbb{P}\left(\lambda_{max} \le \sqrt{2} + \frac{x}{\sqrt{2}n^{2/3}}\right) = F_{GUE}(x).$$

F_{GUE} is defined as [75]

$$(1.17) \qquad F_{GUE}(x) = \det\left(I - A_x\right)$$

of the operator A_x on $L^2(x, \infty)$ with the kernel given in terms of the Airy function Ai by

$$(1.18) \qquad A_x(s,t) = \frac{Ai(s)Ai'(t) - Ai'(s)Ai(t)}{s - t}.$$

It can also be given as an integral [75, 76]

$$(1.19) \qquad F_{GUE}(x) = e^{-\int_x^\infty (s-x)^2 q(s)^2 ds}$$

where q is the so-called Hastings-McLeod solution to the Painlevé II equation $q'' = 2q^3 + xq$ ([45, 41]).

It turns out that F_{GUE} is one of the universal distributions in random matrix theory and also other related areas. Even if we replace the weight function $e^{-n\lambda^2}$ by other general functions $e^{-nV(\lambda)}$, the limiting fluctuation of the largest eigenvalue does not change generically [36, 33]. Wigner matrices, the random Hermitian matrices with i.i.d. entries, also exhibit universality to F_{GUE} [70, 74, 39, 62]. Moreover, F_{GUE} also appears in models outside random matrix theory, such as random permutations [9], directed last passage percolations [50], random growth models [64], non-intersecting random walks [51], asymmetric simple exclusion process [78], etc. These models in statistical physics belong to the so-called KPZ class (see, e.g., [55, 27]), which is believed to have the property that a certain observable fluctuating with a scaling exponent $1/3$. It remains as a challenging problem to prove the universality of F_{GUE} in the general KPZ class.

Let us now consider a time-dependent generalization of GUE. A natural way is to replace the entries of the GUE by Brownian motions. In this case the induced process of the eigenvalues is called the Dyson process. In the large n limit, the process of the largest eigenvalue converges, after appropriate centering and scaling in both time and space, to a limiting process. This limiting process has explicit finite dimensional distributions in terms of a determinant involving the Airy function, and is called the Airy process. The marginal of this Airy process at any given time is the Tracy-Widom distribution. Just like F_{GUE} is a universal limiting distribution of random matrices and random growth models, the Airy process is also a universal limit process. It appears in the polynuclear growth model [65], tiling models [53], the totally asymmetric simple exclusion process (TASEP) [52], the last passage percolation [52], and etc.

The Airy process also arises in nonintersecting processes. It was shown by Dyson that the Dyson process is equivalent to n Brownian motions, all starting at 0 at time 0, subject to the condition that they do not intersect for all time. As such, the Airy process also arises as an appropriate limit of many nonintersecting processes such as Brownian bridges [1], Brownian excursions [77], symmetric simple random walk [51], and etc. Let $X_i(t)$, $i = 1, \cdots, n$, be independent standard Brownian bridges conditioned that $X_1(t) < X_2(t) < \cdots < X_n(t)$ for all $t \in (0, 1)$ and $X_i(0) = X_i(1) = 0$ for all $i = 1, \cdots, n$. It is known that as $n \to \infty$, the top path $X_n(t)$ converges to the curve $x = 2\sqrt{nt(1-t)}$, $0 \le t \le 1$, and the fluctuation around the curve in an appropriate scaling is given by the Airy process $\mathcal{A}(\tau)$ [65]. Especially near the peak location we have (see e.g. [52], [1])

(1.20)
$$2n^{1/6}\left(X_n\left(\frac{1}{2} + \frac{2\tau}{n^{1/3}}\right) - \sqrt{n}\right) \to \mathcal{A}(\tau) - \tau^2$$

in the sense of finite distribution.

In this dissertation we compute the limiting distribution of the so-called width of nonintersecting processes by using discrete Toeplitz determinants. Let $X(t)$ ($0 \leq t \leq T$) be a random process. Consider n processes $(X_1(t), X_2(t), \cdots, X_n(t))$ where $X_i(t)$ is an independent copy of $X(t)$, conditioned that (i) all the X_i starts from the origin and ends at a fixed position, and (ii) $X_1(t) < X_2(t) < \cdots < X_n(t)$ for all $t \in (0, T)$.

Define the width of non-intersecting processes by

$$(1.21) \qquad W_n(T) := \sup_{0 \leq t \leq T} (X_n(t) - X_1(t)).$$

In this dissertation, we first show that the distribution function $W_n(T)$ can be computed explicitly in terms of discrete Toeplitz determinants. We then analyze the asymptotics by using the method indicated in the previous section. The limiting distribution of $W_n(T)$, after rescaling, is exact the Tracy-Widom distribution F_{GUE}.

Combined with (1.20), this result gives rise to an interesting identity between the Airy process and F_{GUE} as follows. Since $\mathcal{A}(\tau)$ is stationary [65], it is reasonable to expect that $\mathcal{A}(\tau) - \tau^2$ is small when $|\tau|$ becomes large, and that the width will be obtained near the time $t = \frac{T}{2}$. Moreover, intuitively the top curve $X_n(t)$ and bottom curve $X_1(t)$ near $t = \frac{T}{2}$ will become far away to each other. Therefore heuristically the two curves near $t = \frac{T}{2}$ are asymptotically independent when n becomes large [1]. These heuristical arguments together with (1.20) suggest that the distribution of the sum of two independent Airy processes is F_{GUE}. More explicitly we have

$$(1.22) \qquad \mathbb{P}\left(\sup_{\tau \in \mathbb{R}} \left(\hat{\mathcal{A}}^{(1)}(\tau) + \hat{\mathcal{A}}^{(2)}(\tau)\right) \leq 2^{1/3}x\right) = F_{GUE}(x),$$

where $\hat{\mathcal{A}}^{(1)}(\tau)$ and $\hat{\mathcal{A}}^{(2)}(\tau)$ are two independent copies of the modified Airy process $\hat{\mathcal{A}}(\tau) := \mathcal{A}(\tau) - \tau^2$. A different identity of similar favor was previously obtained by

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[1]The asymptotical independence of two variables $X_n(t)$ and $X_1(t)$ at $t = \frac{T}{2}$ for the nonintersecting Brownian bridges as n tends to infinity is equivalent to the asymptotical indepedence of the extreme eigenvalues of GUE, which was proved in [20].

Johansson in [52]

$$(1.23) \qquad \mathbb{P}\left[2^{2/3}\sup_{\tau \in \mathbb{R}}\hat{\mathcal{A}}(\tau) \leq x\right] = F_{GOE}(x),$$

where $F_{GOE}(x)$ is an analogue of F_{GUE} for real symmetric matrices. It is natural to ask if there are more such identities. In Chapter V, we prove 5 more such identities.

1.3 Outline of Thesis

In Chapter II, we first review the connection between Toeplitz determinants and orthogonal polynomials. We then discuss a simple identity between discrete Toeplitz determinants and continuous orthogonal polynomials and how this can be used for the asymptotics. This idea is applied to the width of nonintersecting processes in Chapter III and Chapter IV.

In Chapter III, we consider the width of non-intersecting processes whose starting points are same as the ending points. We show that the distribution of width can be represented in terms of discrete Toeplitz determinants. In this case, the associated discrete measure is real-valued. The asymptotics of these discrete Toeplitz determinants is obtained by using the idea developed in Chapter II.

When the ending points of non-intersecting paths are not same as the starting points, then the associated discrete measure is complex-valued. In this case, the asymptotic analysis becomes significantly more difficult. In Chapter IV, we consider one such example, and study the asymptotics of associated discrete Toeplitz determinants. Since there is no general method for the asymptotics of orthogonal polynomials with respect to complex discrete measure, this example should give us new insight to this challenging question.

Finally in chapter V, we prove several identities involving the Airy process and the Tracy-Widom distribution similar to (1.20).

CHAPTER II

Discrete Toeplitz Determinant

2.1 Toeplitz Determinant, Orthogonal Polynomials and Deift-Zhou Steepest Descent Method

We first review a basic relationship between Toeplitz determinants and orthogonal polynomials.

Assume that $f(z)$ is a positive function defined on the unit circle Σ. We define $p_k(z) = \kappa_k z^k + \cdots$ to be the orthogonal polynomials with respect to $f(z)\frac{dz}{2\pi i z}$ which satisfies the following orthogonal conditions:

$$(2.1) \qquad \int_\Sigma p_k(z)\overline{p_j(z)}f(z)\frac{dz}{2\pi i z} = \delta_j(k),$$

where $\delta_j(k)$ is the Dirac delta function, and $j, k = 0, 1, \cdots$. To ensure the uniqueness of $p_k(z)$, we require $\kappa_k > 0$ for all k.

One can construct $p_k(z)$ directly via Toeplitz determinants with symbol f:

$$(2.2) \qquad p_k(z) = \frac{1}{\sqrt{D_k(f)D_{k+1}(f)}} \det \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{-k} \\ f_1 & f_0 & \cdots & f_{-k+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k-1} & f_{k-2} & \cdots & f_{-1} \\ 1 & z & \cdots & z^k \end{pmatrix}, k \geq 1,$$

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and $p_0(f) = \frac{1}{\sqrt{\mathcal{T}_1(f)}}$. Hence $\kappa_k^2 = \frac{\mathcal{T}_k(f)}{\mathcal{T}_{k+1}(f)}$, and $\mathcal{T}_n(f) = \prod_{k=0}^{n-1} \kappa_k^{-2}$. If $\mathcal{T}_\infty = \lim_{n\to\infty} \mathcal{T}_n(f)$ is finite, then we can also express $\mathcal{T}_n(f) = \mathcal{T}_\infty(f)^{-1} \prod_{k=n}^{\infty} \kappa_k^2$. These formulas provide one way to obtain the asymptotics of $\mathcal{T}_n(f)$ via the asymptotics of corresponding orthogonal polynomials.

Remark II.1. Even if we know the asymptotics of κ_k for all k, it could still be complicated to estimate $\mathcal{T}_n(f)$ for n in certain region, where $\log(\kappa_k)$ has polynomial type decay. For example, if we consider the asymptotics of $\mathcal{T}_n(f)$ as $n, t \to \infty$ when $f(z) = e^{t(z+z^{-1})}$, then we need the asymptotics of orthogonal polynomials for all large parameters t and n. It is known [9] that $|\kappa_k^2 - 1| = O(e^{-ck})$ when $2t/k \leq 1 - \delta_1$, and that $\kappa_k^2 = e^{k-2t}(2t/k)^{k-\frac{1}{2}}(1 + O(k^{-1}))$ when $2t/k \geq 1 + \delta_2$, where δ_1, δ_2 are both positive constants. Since $|\log \kappa_k^2|$ diverges for the second regime, the sum of $\log \kappa_k^2$ may be complicated if we want to evaluate the asymptotics to the constant term for some double scaling limits of n and t.

The research on the asymptotics of orthogonal polynomials can be traced back to the 19th century. See [72] for an overview. A powerful method for the study of asymptotics of orthogonal polynomials with respect to a general weight f was developed in the 1990's using the theory of Riemann-Hilbert problems. The formulation of orthogonal polynomials in terms of Riemann-Hilbert problem was discovered by Fokas, Its, and Kitaev in [42]. This formulation was first obtained for orthogonal polynomials on $\mathbb{R}$, but it can be easily adopted to the orthogonal polynomials on Σ. Considered a 2×2 matrix $Y(z)$ which satisfies the following conditions:

- $Y(z)$ is analytic on $\mathbb{C} \setminus \Sigma$.

- $Y(z)z^{-n\sigma_3} = I + O(z^{-1})$ as $z \to \infty$. Here $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n}f(z) \\ 0 & 1 \end{pmatrix}$, for all $z \in \Sigma$. Here $Y_\pm(z) = \lim_{\epsilon \downarrow 0} Y((1 \pm \epsilon)z)$.

To find $Y(z)$ satisfying the above conditions is a matrix-valued Riemann-Hilbert problem. For this specific problem, the solution is given in terms of orthogonal polynomials

$$(2.3) \qquad Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} \int_\Sigma \frac{p_n(s)}{s-z} \frac{f(s)ds}{2\pi i s^n} \\ -\kappa_{n-1} p_{n-1}^*(z) & -\kappa_{n-1} \int_\Sigma \frac{p_{n-1}^*(s)}{s-z} \frac{f(s)ds}{2\pi i s^n} \end{pmatrix},$$

where $p_{n-1}^*(z) = z^n \overline{p_{n-1}(\bar{z}^{-1})}$. Therefore, if we obtain the asymptotics of $Y(z)$ from the Riemann-Hilbert problem, we then can obtain the asymptotics of the orthogonal polynomials.

Deift and Zhou developed a method to obtain the asymptotics of Riemann-Hilbert problems. This method was further extendedand was applied to the Riemann-Hilbert Problems for orthogonal polynomials in [36, 35]. The key idea is to find a contour such that the algebraically-equivalent jump matrix becomes asymptotically a constant matrix on this contour and asymptotically identity matrix elsewhere. By solving the limiting (simpler) Riemann-Hilbert problem explicitly, one may obtain the asymptotics of $Y(z)$ as n becomes large. For the Riemann-Hilbert problem for orthogonal polynomials when f is analytic on Σ, if we use the notation of the so-called g-function [37, 32], one can show that

$$(2.4) \qquad Y(z) = e^{-nl\sigma_3/2} m_\infty(z) e^{nl\sigma_3/2} e^{ng(z)\sigma_3}(1 + o(1)),$$

for z away from Σ, where l is a constant and $m_\infty(z)$ is the solution to the deformed Riemann-Hilbert problem with constant jump. One can further find the error terms explicitly. See [32] for the more details.

Hankel determinant is an analog of Toeplitz determinant. If $f(x)$ is an integrable function on $\mathbb{R}$ such that $\int_\mathbb{R} |x^k f(x)| dx < \infty$ for $k = 0, 1, \cdots$. The n-th Hankel

determinant with *symbol* f is defined to be

$$(2.5) \qquad \mathcal{H}_n(f) = \det\left(\int_{\mathbb{R}} x^{j+k} f(x) dx\right)_{j,k=0}^{n-1}.$$

Similarly to the Toeplitz determinant, we can define the orthogonal polynomials $p_k(x) = \kappa_k x^k + \cdots$ with respect to $f(x)dx$ which satisfies the following orthogonal conditions

$$(2.6) \qquad \int_{\mathbb{R}} p_k(x) p_j(x) f(x) dx = \delta_j(k),$$

for all $j, k = 0, 1, \cdots$. Again we require $\kappa_k > 0$. If $\mathcal{H}_n(f) > 0$ for all $n \geq 0$, one can show the existence and uniqueness of $p_k(x)$. It can be expressed as

(2.7)

$$p_k(x) = \frac{1}{\sqrt{\mathcal{H}_k(f)\mathcal{H}_{k+1}(f)}} \det \begin{pmatrix} \int_{\mathbb{R}} f(x)dx & \int_{\mathbb{R}} x f(x)dx & \cdots & \int_{\mathbb{R}} x^k f(x)dx \\ \int_{\mathbb{R}} x f(x)dx & \int_{\mathbb{R}} x^2 f(x)dx & \cdots & \int_{\mathbb{R}} x^{k+1} f(x)dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathbb{R}} x^{k-1} f(x) & \int_{\mathbb{R}} x^k f(x)dx & \cdots & \int_{\mathbb{R}} x^{2k-1} f(x)dx \\ 1 & x & \cdots & x^k \end{pmatrix}$$

for $k \geq 1$ and $p_0(x) = \frac{1}{\sqrt{\mathcal{H}_1(f)}}$. As a result, $\kappa_k^2 = \frac{\mathcal{H}_k(f)}{\mathcal{H}_{k+1}(f)}$, and $\mathcal{H}_n(f) = \prod_{k=0}^{n-1} \kappa_k^{-2} = \mathcal{H}_\infty(f) \prod_{k=n}^{\infty} \kappa_k^2$.

The Deift-Zhou steepest descent method still works for the asymptotics of orthogonal polynomials on the real line.

Example II.2. If $f(x) = e^{-x^2}$, $p_k(x)$ is the *Hermite polynomial* of degree k, which is one of the most well-known orthogonal polynomials. The asymptotics of Hermite polynomials can be obtained directly by using the usual steepest descent method since it has an integral representation. It can also be obtained by using Deift-Zhou steepest descent method, see [32] for details.

In the end, we define the generalized Toeplitz/Hankel determinants and corresponding orthogonal polynomials. Suppose $\mathfrak{C}$ is a finite union of contours which do not pass the origin and f is a (complex-valued) function on $\mathfrak{C}$ such that

$$(2.8) \qquad \int_{\mathfrak{C}} z^k f(z) \frac{dz}{2\pi i z}$$

exists. Then define

$$(2.9) \qquad \mathcal{T}_n(f, \mathfrak{C}) := \det \left(\int_{\mathfrak{C}} z^{-j+k} f(z) \frac{dz}{2\pi i z} \right)_{j,k=0}^{n-1}$$

for $n \geq 1$. Note that this generalized Toeplitz determinant becomes the usual Toeplitz determinant $\mathcal{T}_n(f)$ defined in (1.1) when $\mathfrak{C}$ is the unit circle.

The orthogonal polynomials $p_k(z)$, $\tilde{p}_k(z)$ ($k = 0, 1, \cdots, n$) with respect to $f(z)\frac{dz}{2\pi i z}$ on $\mathfrak{C}$ are defined as follows. Let $p_k(z), \tilde{p}_k(z)$ be polynomials with degree k which satisfy the following orthogonal conditions:

$$(2.10) \qquad \int_{\mathfrak{C}} p_k(z)\tilde{p}_j(z^{-1})f(z)\frac{dz}{2\pi i z} = \delta_k(j)$$

for all $k, j = 0, 1, \cdots, n$. Such polynomials exist and are unique up to a constant factor provided $\mathcal{T}_k(f, \mathfrak{C}) \neq 0$ for $1 \leq k \leq n + 1$. These orthogonal polynomials also have the three-term recurrence relations and the following Christoffel-Darboux formula (see [34] for more details)

$$(2.11) \qquad \sum_{i=0}^{n-1} p_i(z)\tilde{p}_i(w) = \frac{z^n w^n p_n(w^{-1})\tilde{p}_n(z^{-1}) - p_n(z)\tilde{p}_n(w)}{1 - zw}$$

for all $z, w \in \mathbb{C}$. Furthermore, the following relation between these orthogonal polynomials and the generalized Toeplitz determinants hold: $\mathcal{T}_n(f, \mathfrak{C}) = \prod_{k=0}^{n-1} \kappa_k \tilde{\kappa}_k$.

The generalized Hankel determinant is defined similarly

$$(2.12) \qquad \mathcal{H}_n(f, \mathfrak{C}) = \det \left(\int_{\mathfrak{C}} z^{j+k} dz \right)_{j,k=0}^{n-1}.$$

The corresponding orthogonal polynomials $p_k(z)$ ($k = 0, 1, \cdots, n$) satisfy the following orthogonal conditions:

$$(2.13) \qquad \int_{\mathfrak{C}} p_j(z) p_k(z) f(z) dz = \delta_k(j)$$

for all $k, j = 0, 1, \cdots, n$. Such orthogonal polynomials exist and are unique up to the factor -1 provided $\mathcal{H}_k(f, \mathfrak{C}) \neq 0$ for $1 \leq k \leq 2n + 2$. It is easy to see that the three-term recurrence relations and Christoffel-Darboux formula still hold and the proofs are exact the same as that for the orthogonal polynomials on the real line.

2.2 Discrete Toeplitz/Hankel Determinant and Discrete Orthogonal Polynomials

It is natural to ask the analogous case when the measure is discrete. More explicitly, let $\mathcal{D}$ be a countable discrete set on $\mathbb{C}$. Suppose f is a function on $\mathcal{D}$. Define the discrete Toeplitz/Hankel determinant with measure $\sum_{z \in \mathcal{D}} f(z)$ as

$$(2.14) \qquad \mathrm{T}_n(f, \mathcal{D}) := \det \left(\sum_{z \in \mathcal{D}} z^{-j+k} f(z) \right)_{j,k=0}^{n-1},$$

$$(2.15) \qquad \mathrm{H}_n(f, \mathcal{D}) := \det \left(\sum_{z \in \mathcal{D}} z^{j+k} f(z) \right)_{j,k=0}^{n-1}.$$

We emphasize two important features in addition to discreteness of the associated measure. The first is that the support of the measure is not necessarily a part of the unit circle (or real line). The second is that the measure may be complex-valued. These two changes do not affect the algebraic formulation much, but significantly increase the difficulty of the asymptotic analysis.

Similarly to the continuous Toeplitz/Hankel determinants, one can find the relation between discrete Toeplitz/Hankel determinants and *discrete orthogonal polynomial.*

For discrete Toeplitz determinants, one introduces the discrete orthogonal polynomials as follows. Let $p_k(z) = \kappa_k z^k + \cdots$ and $\tilde{p}_k(z) = \tilde{\kappa}_k z^k + \cdots$ be the orthogonal polynomials with respect to the discrete measure $\sum_{z \in \mathcal{D}} f(z)$ which satisfy the following orthogonal condition

$$(2.16) \qquad \sum_{z \in \mathcal{D}} p_k(z) \tilde{p}_j(z^{-1}) f(z) = \delta_j(k)$$

for all $j, k = 0, 1, \cdots, n$. If $\mathrm{T}_k(f, \mathcal{D}) \neq 0$ for all $1 \leq k \leq n$, one can show the orthogonal polynomials exist and are unique up to a constant factor.

Remark II.3. When $f > 0$ and $\mathcal{D}$ is a subset of the unit circle, then it is a direct to check $\tilde{p}_k(z) = \overline{p_k(\bar{z})}$.

Similarly to the continuous orthogonal polynomials (2.2), one can construct the discrete orthogonal polynomials $p_k(z)$ and $\tilde{p}_k(z)$ as following:

$$(2.17) \qquad p_k(z) = \frac{1}{\sqrt{\mathrm{T}_k(f, \mathcal{D}) \mathrm{T}_{k+1}(f, \mathcal{D})}} \det \begin{pmatrix} \sum_{z \in \mathcal{D}} f(z) & \sum_{z \in \mathcal{D}} z f(z) & \cdots & \sum_{z \in \mathcal{D}} z^k f(z) \\ \sum_{z \in \mathcal{D}} z^{-1} f(z) & \sum_{z \in \mathcal{D}} f(z) & \cdots & \sum_{z \in \mathcal{D}} z^{k-1} f(z) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{z \in \mathcal{D}} z^{-k+1} f(z) & \sum_{z \in \mathcal{D}} z^{-k+2} f(z) & \cdots & \sum_{z \in \mathcal{D}} z f(z) \\ 1 & z & \cdots & z^k \end{pmatrix},$$

$$(2.18) \qquad \tilde{p}_k(z^{-1}) = \frac{1}{\sqrt{\mathrm{T}_k(f, \mathcal{D}) \mathrm{T}_{k+1}(f, \mathcal{D})}} \det \begin{pmatrix} \sum_{z \in \mathcal{D}} f(z) & \sum_{z \in \mathcal{D}} z^{-1} f(z) & \cdots & \sum_{z \in \mathcal{D}} z^{-k} f(z) \\ \sum_{z \in \mathcal{D}} z f(z) & \sum_{z \in \mathcal{D}} f(z) & \cdots & \sum_{z \in \mathcal{D}} z^{-k+1} f(z) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{z \in \mathcal{D}} z^{k-1} f(z) & \sum_{z \in \mathcal{D}} z^{k-2} f(z) & \cdots & \sum_{z \in \mathcal{D}} z^{-1} f(z) \\ 1 & z^{-1} & \cdots & z^{-k} \end{pmatrix},$$

for all $1 \leq k \leq n$, and $p_0(z) = \tilde{p}_0(z) = \frac{1}{\sqrt{T_1(f,\mathcal{D})}}$.

Therefore one can write $T_n(f, \mathcal{D}) = \prod_{k=0}^{n-1} \kappa_k^{-1} \tilde{\kappa}_k^{-1}$.

In [12], the authors extended the Deift-Zhou steepest descent method to the discrete orthogonal polynomials when $\mathcal{D} \subset \mathbb{R}$. For the case when $\mathcal{D} \subset \Sigma$, we have the following. Define

$$(2.19) \qquad Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} \sum_{s \in \mathcal{D}} \frac{p_n(s)}{s-z} \frac{f(s)}{s^{n-1}} \\ Y_{21}(z) & \sum_{s \in \mathcal{D}} \frac{Y_{21}(s)}{s-z} \frac{f(s)}{s^{n-1}} \end{pmatrix},$$

where $Y_{21}(z)$ is given by

$$(2.20)$$

$$\frac{(-1)^n}{T_n(f,\mathcal{D})} \det \begin{pmatrix} \sum_{z \in \mathcal{D}} z^{-1} f(z) & \sum_{z \in \mathcal{D}} f(z) & \cdots & \sum_{z \in \mathcal{D}} z^{n-2} f(z) \\ \sum_{z \in \mathcal{D}} z^{-2} f(z) & \sum_{z \in \mathcal{D}} z^{-1} f(z) & \cdots & \sum_{z \in \mathcal{D}} z^{n-3} f(z) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{z \in \mathcal{D}} z^{-n+1} f(z) & \sum_{z \in \mathcal{D}} z^{-n+2} f(z) & \cdots & \sum_{z \in \mathcal{D}} f(z) \\ 1 & z & \cdots & z^{n-1} \end{pmatrix}.$$

Then one can show that $Y(z)$ is the unique solution to the Riemann-Hilbert problem which has the following requirements:

- $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \mathcal{D}$.

- $Y(z) z^{-n\sigma_3} = I + O(z^{-1})$ as $z \to \infty$.

- At each node $z \in \mathcal{D}$, the first column of $Y(z)$ is analytic and the second column of $Y(z)$ has a simple pole, where the residue satisfies the condition

$$(2.21) \qquad \operatorname{Res}_{z'=z} Y(z') = \lim_{z' \to z} Y(z') \begin{pmatrix} 0 & -\frac{f(z')}{z'^{n-1}} \\ 0 & 0 \end{pmatrix}.$$

Similarly, one can find the corresponding Riemann-Hilbert problem for $\tilde{p}_n(z)$.

It is of great interests to solve this type of *discrete Riemann-Hilbert problem* asymptotically. Consider the case when $\mathcal{D}$ is a subset of the unit circle and $f(z) =$

$e^{-nV(z)}$ where $V(z)$ is a real function on the unit circle. If we ignore the discreteness, there is a so-called *equilibrium measure* $d\mu_0(z)$ such that the following *energy function* reaches its minimal at μ_0

$$(2.22) \qquad E(\mu) := -\int_\Sigma \int_\Sigma \log|z - w| d\mu(z) d\mu(w) + \int_\Sigma V(z) d\mu(z).$$

The g-function for the corresponding Riemann-Hilbert problem can be constructed by using this equilibrium measure. And the asymptotics of the continuous orthogonal polynomials will also be relevant to μ_0. One can see the relation heuristically as following. By using (2.2) one can write

$$(2.23) \qquad p_n(z) = C_n \int_{\Sigma^n} e^{2\sum_{1 \le i < j \le n} |z_i - z_j| - n\sum_{i=1}^n V(z_i) + \sum_{i=1}^n \log(z - z_i)} \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_n}{2\pi i z_n},$$

therefore heuristically one may expect that

$$(2.24) \qquad p_n(z) \sim C_n e^{-n^2 E(\mu_0) + n\int_\Sigma \log(z - s) d\mu_0(s)}.$$

Now we take the discreteness into consideration. This condition will give a so-called *upper constraint* on the equilibrium measure, which requires that the measure μ_0 is bounded above by the counting measure $|\mathcal{D}|^{-1}\sum_{z \in \mathcal{D}} \delta_z$. This restrictions can be heuristically seen in the discrete version of (2.23) where z_i's are selected from the nodes set $\mathcal{D}$. In [12], the authors systematically discussed this upper constraint issue for discrete discrete orthogonal polynomials on the real line $\mathbb{R}$. They remove the poles and deform the corresponding discrete Riemann-Hilbert problem to a usual Riemann-Hilbert problem with jump contours. Once the upper constraint condition is triggered, the g-function will has a so-called *saturated region*. By deforming the Riemann-Hilbert problem accordingly one will still be able to obtain the asymptotics of $Y(z)$ when the parameters go to infinity simultaneously. Their method is also believed to work for the upper constraint issue on the unit circle Σ.

As a result, for the discrete Toeplitz determinant with positive symbol f and $\mathcal{D}$ is a subset of the unit circle, one can find the asymptotics of the discrete orthogonal polynomials. Furthermore, it is possible to find the asymptotics of $\mathrm{T}_n(f, \mathcal{D})$ by using that of discrete orthogonal polynomials. However, there are some limitations of this approach:

First, even if the upper constraint is inactive, the asymptotics of the discrete orthogonal polynomials will have the same leading term with that of the corresponding continuous orthogonal polynomials. Therefore one would expect some complications in summarizing $\log \kappa_k$'s in certain parameter region, as we mentioned in the Remark II.1. We will see from Theorem II.6 that these complicities come from the continuous counterpart of $\mathrm{T}_n(f, \mathcal{D})$.

Second, if the upper constraint is active, we will have new complications coming from the saturated region. In this case, the leading terms of the discrete orthogonal polynomials will be different from the continuous orthogonal polynomials. It is not clear whether one can summarize $\log \kappa_k$'s for these k's.

Finally, in some cases we are interested in the asymptotics of $\mathrm{T}_n(f, \mathcal{D})$ when f is not real. In these cases we do not have a good understanding of the upper constraint or equilibrium measure, hence it is not clear how to apply the techniques of the saturated region of the equilibrium measure to the corresponding discrete orthogonal polynomials.

For the discrete Hankel determinant $\mathrm{H}_n(f, \mathcal{D})$ we can similarly define the orthogonal polynomials and construct the corresponding discrete Riemann-Hilbert problem. And we will have similar limitations to find the asymptotics of $\mathrm{H}_n(f, \mathcal{D})$ by using that of discrete orthogonal problems, as we discussed above.

2.3   A Simple Identity on the Discrete Toeplitz/Hankel Determinant

One of the main results of this dissertation is a simple identity on the discrete Toeplitz/Hankel determinant. This identity expresses the discrete Toeplitz/Hankel determinant as the product of a continuous Toeplitz/Hankel determinant and a Fredholm determinant, as explained below.

We first consider the case of discrete Toeplitz determinant. The case of discrete Hankel determinant will be stated in the end. To state the identity, let Ω be a neighborhood of $\mathcal{D}$. Suppose $\gamma(z)$ be a function which is analytic in Ω and $\mathcal{D} = \{z \in \Omega | \gamma(z) = 0\}$. Moreover, all these roots are simple. Note that the existence of γ is guaranteed by Weierstrass factorization theorem.[1] We assume the followings:

(a) $f(z)$ can be extended to an analytic function in Ω. We still use the notation $f(z)$ for this analytic function.

(b) There exists a finite union of oriented contours $\mathfrak{C}$ in Ω, such that $0 \notin \mathfrak{C}$ and

$$(2.25) \qquad \int_{\mathfrak{C}} \frac{\gamma'(z)}{2\pi i \gamma(z)} z^k f(z) dz = \sum_{z \in \mathcal{D}} z^k f(z)$$

for all $|k| \leq n - 1$.

(c) There exists a function $\rho(z)$ on $\mathfrak{C}$ such that the (generalized) Toeplitz determinants with symbol $f\rho$

$$(2.26) \qquad \mathcal{T}_k(f\rho, \mathfrak{C}) = \det \left(\int_{\mathfrak{C}} z^{-i+j} f(z) \rho(z) \frac{dz}{2\pi i z} \right)_{i,j=0}^{l-1}$$

exists and is nonzero, for all $1 \leq k \leq n$.

Remark II.4. When $\mathcal{D}$ is a subset of the unit circle Σ, one can choose $\mathfrak{C}$ to be the union of the following two circles both centered at the origin. One is of radius $1 + \epsilon$ and is oriented in counterclockwise direction. The other one is of radius $1 - \epsilon$ and is

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[1] If $\mathcal{D} = \{z_1, \cdots, z_m\}$ is a finite set, one can define $\gamma(z) = \prod_{i=1}^{m}(z - z_i)$ which is a polynomials.

oriented in clockwise direction. Here $\epsilon > 0$ is a constant such that $f(z)$ is analytic within the region enclosed by $\mathfrak{C}$. Then (b) is automatically satisfied by a residue computation.

Remark II.5. If $\rho(z)$ is analytic in a neighborhood of $\mathfrak{C}$, the continuous Toeplitz determinants (2.26) and the corresponding orthogonal polynomials are independent of the choice of $\mathfrak{C}$.

Now we are ready to state the main Theorem.

Theorem II.6. *Under the assumptions (a), (b) and (c) above, we have*

$$(2.27) \qquad \mathrm{T}_n(f, \mathcal{D}) = \mathcal{T}_n(f\rho, \mathfrak{C}) \det(I + K)$$

where $\det(I + K)$ *is a Fredholm determinant defined by*

$$(2.28) \quad \det(I + K) := 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \int_{\mathfrak{C}} \cdots \int_{\mathfrak{C}} \det\left(K(z_j, z_k)\right)_{j,k=0}^{l-1} \frac{dz_0}{2\pi i z_0} \cdots \frac{dz_{l-1}}{2\pi i z_{l-1}},$$

K *is an integral operator with kernel*

$$(2.29)$$
$$K(z, w) = \sqrt{v(z)v(w)f(z)f(w)} \frac{(z/w)^{\frac{n}{2}} p_n(w)\tilde{p}_n(z^{-1}) - (w/z)^{\frac{n}{2}} p_n(z)\tilde{p}_n(w^{-1})}{1 - zw^{-1}},$$

$p_n(z)$, $\tilde{p}_n(z)$ *are orthogonal polynomials with respect to* $f(z)\rho(z)\frac{dz}{2\pi i z}$ *on* $\mathfrak{C}$, *as we defined at the end of Section 2.1, and*

$$(2.30) \qquad v(z) := \frac{z\gamma'(z)}{\gamma(z)} - \rho(z).$$

Proof. We first use (2.25) and write

$$\mathrm{T}_n(f, \mathcal{D}) = \det\left(\int_{\mathfrak{C}} \frac{\gamma'(z)}{2\pi i \gamma(z)} z^{-j+k} f(z) dz\right)_{j,k=0}^{n-1}$$

$$(2.31) \qquad = \frac{1}{\prod_{k=0}^{n-1} \kappa_k \tilde{\kappa}_k} \det\left(\int_{\mathfrak{C}} \frac{\gamma'(z)}{2\pi i \gamma(z)} p_k(z)\tilde{p}_j(z^{-1}) f(z) dz\right)_{j,k=0}^{n-1}$$

$$= \frac{1}{\prod_{k=0}^{n-1} \kappa_k \tilde{\kappa}_k} \det\left(\delta_j(k) + \int_{\mathfrak{C}} p_k(z)\tilde{p}_j(z^{-1}) v(z) f(z) \frac{dz}{2\pi i z}\right)_{j,k=0}^{n-1},$$

where we performed the row/column operations in the second equation and used the orthogonal conditions of $p_k(z), \tilde{p}_j(z)$ in the third equation. Now we use the well-known identity $\det(I+AB) = \det(I+BA)$ where A is an operator from $L^2(\Sigma, \frac{dz}{2\pi i z})$ to $l^2(\{0, \cdots, n-1\})$ with kernel $A(j, z) := \tilde{p}_j(z^{-1})v(z)f(z)$ and B is an operator from $l^2(\{0, \cdots, n-1\})$ to $L^2(\Sigma, \frac{dz}{2\pi i z})$ with kernel $B(z, k) := p_k(z)$, and the Christoffel-Darboux formula (2.11)

$$(2.32) \qquad \mathrm{T}_n(f, \mathcal{D}) = \frac{1}{\prod_{k=0}^{n-1} \kappa_k \tilde{\kappa}_k} \det\left(I + \tilde{K}\right)$$

where

$$(2.33) \qquad \tilde{K}(z, w) = v(z)f(z)\frac{(z/w)^n p_n(w)\tilde{p}_n(z^{-1}) - p_n(z)\tilde{p}_n(w^{-1})}{1 - zw^{-1}}.$$

By applying $\mathcal{T}_n(f\rho, \mathfrak{C}) = \frac{1}{\prod_{k=0}^{n-1} \kappa_k \tilde{\kappa}_k}$ and a conjugation on the determinant, we have

$$(2.34) \qquad \mathrm{T}_n(f, \mathcal{D}) = \mathcal{T}_n(f\rho, \mathfrak{C}) \det(1 + K).$$

$\square$

Similarly for the discrete Hankel determinant $\mathrm{H}_n(f, \mathcal{D})$, let Ω be a neighborhood of $\mathcal{D}$. Suppose $\gamma(z)$ be a function which is analytic in Ω and $\mathcal{D} = \{z \in \Omega | \gamma(z) = 0\}$ is the set of all the roots of γ in Ω. Moreover, all these roots are simple. We assume the followings:

(a) $f(z)$ is a non-trivial analytic function on Ω.

(b) There exists a contour $\mathfrak{C}$ such that (2.25) holds for all $0 \leq k \leq 2n - 2$.

(c) There exists a function $\rho(z)$ on $\mathfrak{C}$ such that the (generalized) Hankel determinants with symbols $f\rho$

$$(2.35) \qquad \mathcal{H}_k(f\rho, \mathfrak{C}) = \det\left(\int_{\mathfrak{C}} z^{i+j} f(z)\rho(z)\right)_{i,j=0}^{k-1}$$

exists and is nonzero, for all $1 \leq k \leq n$.

We have the following analogous result to Theorem II.6.

Theorem II.7. *Under the assumptions (a), (b) and (c) above, then*

$$(2.36) \qquad \mathrm{H}_n(f, \mathcal{D}) = \mathcal{H}_n(f\rho, \mathfrak{C}) \det(I + K)$$

where $\det(I + K)$ *is a Fredholm determinant defined by*

$$(2.37) \qquad \det(I + K) := 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \int_{\mathfrak{C}} \cdots \int_{\mathfrak{C}} \det\left(K(z_j, z_k)\right)_{j,k=0}^{l-1} dz_0 \cdots dz_{l-1},$$

K *is an integral operator with kernel*

$$(2.38) \qquad K(z, w) = \sqrt{v(z)v(w)f(z)f(w)} \frac{\kappa_{n-1}}{\kappa_n} \frac{p_n(z)p_{n-1}(w) - p_{n-1}(z)p_n(w)}{z - w},$$

$p_k(z) = \kappa_k z^k + \cdots$ *is the continuous orthogonal polynomial of degree* k *with respect to* $f(z)\rho(z)dz$ *on* $\mathfrak{C}$, $k = 0, \cdots, n$, *and*

$$(2.39) \qquad v(z) := \frac{\gamma'(z)}{2\pi i \gamma(z)} - \rho(z).$$

(2.27) and (2.36) are two simple identities which relate the discrete Toeplitz/Hankel determinants to the continuous Toeplitz/Hankel determinants. These continuous Toeplitz/Hankel determinants have natural interpretation as follows. Assume that (1) $\mathcal{D} = \mathcal{D}_m$ contains a large parameter m, (2) both $\mathfrak{C}$ and $f\rho$ are independent of m, and (3) $\gamma = \gamma_m$ satisfies

$$(2.40) \qquad \lim_{m \to \infty} \frac{z\gamma'_m(z)}{\rho(z)\gamma_m(z)} = 1$$

for the Toeplitz case or

$$(2.41) \qquad \lim_{m \to \infty} \frac{\gamma'_m(z)}{2\pi i \rho(z)\gamma_m(z)} = 1$$

for the Hankel case, for all $z \in \mathfrak{C}$. Under these three assumptions, we have

$$(2.42) \qquad \mathcal{T}_n(f\rho, \mathfrak{C}) = \lim_{m \to \infty} \mathrm{T}_n(f, \mathcal{D}_m)$$

in Theorem II.6, or

$$(2.43) \qquad \mathcal{H}_n(f\rho, \mathfrak{C}) = \lim_{m \to \infty} \mathrm{H}_n(f, \mathcal{D}_m)$$

in Theorem II.7. Therefore, the continuous Toeplitz/Hankel determinants in Theorem II.6 and Theorem II.6 can be interpreted as the limit of the discrete Toeplitz/Hankel determinants as the parameter m of the nodes set goes to infinity. Moreover, the Fredholm determinant $\det(1 + K)$ can be understood as the ratio of the discrete Toeplitz/Hankel determinant and its continuous counterpart.

Now we discuss an application of these two identities (2.27) and (2.36). They can be used to analyze the asymptotics of discrete Toeplitz/Hankel determinants as all the parameters go to infinity. Compared with the approach of discrete orthogonal polynomials mentioned in the previous section, these two identities have the following advantages.

The most important advantage of this method is that one can use continuous orthogonal polynomials instead of discrete orthogonal polynomials to do the asymptotic analysis of the discrete Toeplitz/Hankel determinants. As discussed in the previous two sections, there is a more developed theory for the asymptotics of continuous orthogonal polynomials than that for discrete orthogonal polynomials. The Deift-Zhou steepest descent method works for a very general class of weight functions $f(z)$, whereas the saturated region argument works for a less general class of weight function. Moreover, even if the weight functions f is nice enough, the asymptotic analysis of continuous orthogonal polynomials is always easier than that of discrete orthogonal polynomials. Considering these two factors, it is a significant simplification to convert the problem to continuous orthogonal polynomials.

The second advantage of this method is that the kernel $K(z, w)$ has a nice structure. It contains a Christoffel-Darboux kernel part which only depends on the con-

tinuous orthogonal polynomials, and a multiplier part which has singularities on $\mathcal{D}$. The Christoffel-Darboux kernel arises naturally in the unitary invariant ensembles of random matrix theory and is well-studied. It turns out to be convergent, after rescaling, to the so-called sine kernel or Airy kernel in unitary invariant ensembles [36, 35]. Therefore the asymptotics of the kernel $K(z, w)$ can be obtained similarly. Moreover, the Christoffel-Darboux structure can be utilized to rewrite the Fredholm determinant $\det(1+K)$ when the original kernel is ineffective for asymptotic analysis or the deformation of integral contours is obstructed by the singularities of K. See Section 4.2.1 for details.

Finally, these two identities give a way to compute the ratio of the discrete Hankel/Toeplitz determinants and its continuous counterpart without computing any of these determinants. Sometimes this ratio has combinatorial meaning and its asymptotics is of interests [14, 59, 13]. It turns out that the asymptotics of this ratio can be computed out even when we do not know or cannot compute the asymptotics of the continuous and discrete Toeplitz/Hankel determinants.

As an application, we will discuss the following four examples of discrete Toeplitz/Hankel determinant:

(a) $\mathrm{H}_n(e^{-nx^2}, \mathcal{D}_M)$, where $\mathcal{D}_M = \{\sqrt{2}\pi M^{-1}n^{-1/2}k | k \in \mathbb{Z}\}$.

(b) $\mathrm{T}_n(e^{\frac{T}{2}(z+z^{-1})}, \mathcal{D}_M)$, where $\mathcal{D}_M = \{z | z^M = 1\}$.

(c) $\mathrm{T}_n(z^{-T}(1+z)^{2T}, \mathcal{D}_M)$, where $\mathcal{D}_M = \{z | z^M = 1\}$.

(d) $\mathrm{T}_n(z^{-a}e^{Tz}, \mathcal{D}_M)$, where $\mathcal{D}_M = \{z^M = r^M\}$ for certain constant r.

All these examples comes from the models of non-intersecting processes. In the first three examples, the weight function is real on the set of nodes. And in the last example, the weight function is complex-valued on the set of nodes. We will illustrate how the identities (2.27) and (2.36) apply to these models.

CHAPTER III

Asymptotics of the Ratio of Discrete Toeplitz/Hankel Determinant and its Continuous Counterpart, the Real Weight Case

In this chapter, we apply the idea discussed in Chapter II to the width of noninter-secting processes of three different types: Brownian bridges, continuous-time simple random walk, discrete time simple random walk. The main part of this chapter was published in [14].

3.1 Nonintersecting Brownian Bridges

Let $X_i(t)$, $i = 1, \cdots, n$, be independent standard Brownian motions conditioned that $X_1(t) < X_2(t) < \cdots < X_n(t)$ for all $t \in (0, 1)$ and $X_i(0) = X_i(1) = 0$ for all $i = 1, \cdots, n$. The width is defined as

$$(3.1) \qquad W_n := \sup_{0 \leq t \leq 1} \left(X_n(t) - X_1(t) \right).$$

Note that the event that $W_n < M$ equals the event that the Brownian motions stay in the chamber $x_1 < x_2 < \cdots < x_n < x_1 + M$ for all $t \in (0, 1)$. An application of the Karlin-McGregor argument in the chamber [56, 46] implies the following formula.

Proposition III.1. *Let W_n be defined in* (3.1). *Then*

$$(3.2) \qquad \mathbb{P}\left(W_n < M \right) = \frac{\left(\frac{\sqrt{2\pi}}{M\sqrt{n}} \right)^n}{\mathcal{H}_n(f)} \int_0^1 \mathrm{H}_n(f, \mathcal{D}_s) ds, \qquad f(x) = e^{-nx^2},$$

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where

$$(3.3) \qquad \mathcal{D}_s := \mathcal{D}_{M,s} = \left\{ \frac{\sqrt{2\pi}}{M\sqrt{n}}(m - s) : m \in \mathbb{Z} \right\}.$$

By using Theorem II.7, the asymptotics of the above probability can be studied by using the continuous orthogonal polynomials with respect to e^{-nx^2}, i.e. Hermite polynomials. We obtain:

Theorem III.2. *Let W_n be the width of n non-intersecting Brownian bridges with duration 1 given in (3.1). Then for every $x \in \mathbb{R}$,*

$$(3.4) \qquad \lim_{n\to\infty} \mathbb{P}\left((W_n - 2\sqrt{n})2^{2/3}n^{1/6} \leq x\right) = F_{GUE}(x).$$

Remark III.3. The discrete Hankel determinant $H_n(F, \mathcal{D}_0)$ with $s = 0$ was also appeared in [44] (see Model I and the equation (14), which is given in terms of a multiple sum) in the context of a certain normalized reunion probability of non-intersecting Brownian motions with periodic boundary condition. In the same paper, a heuristic argument that a double scaling limit is $F(x)$ was discussed. Nevertheless, the interpretation in terms of the width of non-intersecting Brownian motions and a rigorous asymptotic analysis were not given in [44].

3.2 Proof of Proposition III.1

We prove Proposition III.1.

Let $D_n := \{x_0 < x_1 < \cdots < x_{n-1}\} \subset \mathbb{R}^n$. Fix $\alpha = (\alpha_0, \cdots, \alpha_{n-1}) \in D_n$ and $\beta = (\beta_0, \cdots, \beta_{n-1}) \in D_n$. Let $X(t) = (X_0(t), X_1(t), \cdots, X_{n-1}(t))$ be n independent standard Brownian motions. We denote the conditional probability that $X(0) = \alpha$ and $X(1) = \beta$ by $\mathbb{P}_{\alpha,\beta}$. Let $\mathfrak{N}_0$ be the event that $X(t) \in D_n$ for all $t \in (0,1)$ and let $\mathfrak{N}_1$ be the event that $X(t) \in D_n(M) := \{x_0 < x_1 < \cdots < x_{n-1} < x_0 + M\}$. Then $\mathbb{P}(W_n < M)$ may be computed by taking the limit of $\frac{\mathbb{P}_{\alpha,\beta}(\mathfrak{N}_1)}{\mathbb{P}_{\alpha,\beta}(\mathfrak{N}_0)}$ as $\alpha, \beta \to 0$.

From the Karlin-McGregor argument [56], $\mathbb{P}_{\alpha,\beta}(\mathfrak{N}_0) = \frac{\det[p(\alpha_j - \beta_k)]_{j,k=0}^{n-1}}{\prod_{j=0}^{n-1} p(\alpha_j - \beta_j)}$, where $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. On the other hand, the Karlin-McGregor argument in the chamber $D_n(M)$ was given for example in [46] and implies the following. For convenience of the reader, we include a proof.

Lemma III.4. *The probability* $\mathbb{P}_{\alpha,\beta}(\mathfrak{N}_1)$ *equals*

$$(3.5) \qquad \frac{1}{\prod_{j=0}^{n-1} p(\alpha_j - \beta_j)} \sum_{\substack{h_j \in \mathbb{Z} \\ h_0 + h_1 + \cdots + h_{n-1} = 0}} \det\left[p(\alpha_j - \beta_k + h_k M)\right]_{j,k=0}^{n-1}.$$

Proof. For $\beta = (\beta_0, \cdots, \beta_{n-1}) \in D_n(M)$, let $\mathbb{L}_M(\beta)$ be the set of all n-tuples $(\beta_0' + h_0 M, \cdots, \beta_{n-1}' + h_{n-1} M)$ where $(\beta_0', \cdots, \beta_{n-1}')$ is an re-arrangment of $(\beta_0, \cdots, \beta_{n-1})$ and $h_0, \cdots, h_{n-1}$ are n integers of which the sum is 0. The key property of $\mathbb{L}_M(\beta)$ is that $\mathbb{L}_M(\beta) \cap D_n(M) = \{\beta\}$. Indeed note that since $\beta \in D_n(M)$, we have $|\beta_i' - \beta_j'| < M$ for all i, j. Thus if $(\beta_0' + h_0 M, \cdots, \beta_{n-1}' + h_{n-1} M) \in D_n(M)$, then we have $h_0 \leq \cdots \leq h_{n-1} \leq h_0 + 1$. Since $h_0 + \cdots + h_{n-1} = 0$, this implies that $h_0 = \cdots = h_{n-1} = 0$. This implies that $\beta_j' = \beta_j$ for j and $\mathbb{L}_M(\beta) \cap D_n(M) = \{\beta\}$.

Now we consider n independent standard Brownian motions $X(t)$, $0 \leq t \leq 1$, satisfying $X(0) = \alpha$ and $X(1) \in \mathbb{L}_M(\beta)$. Then one of the following two events happens:

(a) $X(t) \in D_n(M)$ for all $t \in [0,1]$. In this case, $X(1) = \beta$.

(b) There exists a smallest time t_{min} such that $X(t_{min})$ is on the boundary of the chamber $D_n(M)$. Then almost surely one of the following two events happens: (b1) a unique pair of two neighboring Brownian motions intersect each other at time t_{min}, (b2) $X_{n-1}(t_{min}) - X_0(t_{min}) = M$. By exchanging the two corresponding Brownian motions after time t_{min} in the case (b1), or replacing $X_0(t), X_{n-1}(t)$ by $X_{n-1}(t) - M, X_0(t) + M$ respectively after time t_{min} in the case (b2), we obtain two new Brownian motions. See Figure 3.1 for an illustration. Define $X^*(t)$ be the
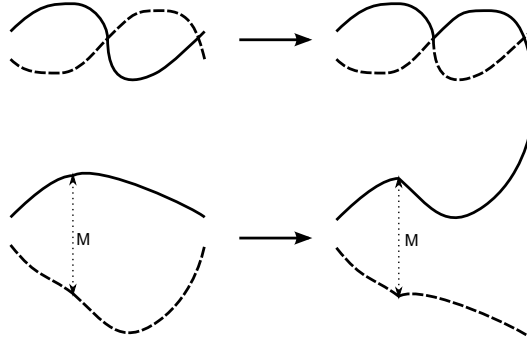
Figure 3.1: Illustration of exchanging two paths

these two new Brownian motions together with the other $n-2$ Brownian motions. Then clearly, $X^*(1) \in \mathbb{L}_M(\beta)$. It is easy to see that $(X^*)^*(t) = X(t)$ and hence this defines an involution on the event (b) almost surely. By expanding the determinant in the sum in (3.5) and applying the involution, we find that that this sum equals the probability that $X(t)$ is from α to β such that $X(t)$ stays in $D_n(M)$. Hence Lemma III.4 follows. $\qquad\square$

Define the generating function

$$(3.6) \qquad g(x,\theta) := \sum_{h \in \mathbb{Z}} p(x+hM)e^{iMh\theta}.$$

It is direct to check that the sum in (3.5) equals $\frac{M}{2\pi} \int_0^{\frac{2\pi}{M}} \det\left[g(\alpha_j - \beta_k, \theta)\right]_{j,k=0}^{n-1} d\theta$.

Thus, we find that

$$(3.7) \qquad \frac{\mathbb{P}_{\alpha,\beta}(\mathfrak{N}_1)}{\mathbb{P}_{\alpha,\beta}(\mathfrak{N}_0)} = \frac{\frac{M}{2\pi} \int_0^{\frac{2\pi}{M}} \det\left[g(\alpha_j - \beta_k, \theta)\right]_{j,k=0}^{n-1} d\theta}{\det\left[p(\alpha_j - \beta_k)\right]_{j,k=0}^{n-1}}.$$

By taking the limit $\alpha, \beta \to 0$, we obtain:

Lemma III.5. *We have*

$$(3.8) \qquad \mathbb{P}\left(W_n < M\right) = \int_0^1 \frac{\left(\frac{\sqrt{2}\pi}{M\sqrt{n}}\right)^n \sum_{x \in \mathcal{D}_{m,s}^n} \Delta(x)^2 \prod_{j=0}^{n-1} e^{-nx_j^2}}{\int_{x \in \mathbb{R}^n} \Delta(x)^2 \prod_{j=0}^{n-1} e^{-nx_j^2} dx_j} ds,$$

where $\mathcal{D}_{m,s} := \left\{ \frac{\sqrt{2}\pi}{M\sqrt{n}}(m-s) : m \in \mathbb{Z} \right\} \subset \mathbb{R}$ and $\Delta(x)$ denotes the the Vandermonde determinant of $x = (x_0, \cdots, x_{n-1})$.

Proof. We insert $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ into (3.6) and then use the Poisson summation formula to obtain

$$(3.9) \qquad g(x,\theta) = \frac{1}{M} \sum_{h \in \mathbb{Z}} e^{-\frac{1}{2}\left(\frac{2\pi h}{M} - \theta\right)^2 + ix\left(\frac{2\pi h}{M} - \theta\right)}.$$

Using the Andreief's formula [4], $\det\left[g(\alpha_j - \beta_k, \theta)\right]_{j,k=0}^{n-1}$ equals

$$(3.10) \qquad \frac{1}{n!M^n} \sum_{h \in \mathbb{Z}^n} \det\left[e^{i\alpha_j\left(\frac{2\pi h_k}{M} - \theta\right)}\right]_{j,k=0}^{n-1} \det\left[e^{-i\beta_j\left(\frac{2\pi h_k}{M} - \theta\right)}\right]_{j,k=0}^{n-1} \prod_{j=0}^{n-1} e^{-\frac{1}{2}\left(\frac{2\pi h_j}{M} - \theta\right)^2}.$$

Since $\det\left[e^{x_j y_k}\right]_{j,k=0}^{n-1} = c\Delta(x)\Delta(y)(1 + O(y))$ with $c = \prod_{j=0}^{n-1} \frac{1}{j!}$ as $y \to 0$ for each x, we find that

$$(3.11) \qquad \lim_{\alpha,\beta \to 0} \frac{\det\left[g(\alpha_j - \beta_k, \theta)\right]_{j,k=0}^{n-1}}{c^2\Delta(\alpha)\Delta(\beta)} = \frac{(2\pi/M)^{n(n-1)}}{n!M^n} \sum_{h \in \mathbb{Z}^n} \Delta(h)^2 \prod_{j=0}^{n-1} e^{-\frac{1}{2}\left(\frac{2\pi h_j}{M} - \theta\right)^2}$$

On the other hand, using $p(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2 + ixy} dy$,

$$(3.12) \qquad \lim_{\alpha,\beta \to 0} \frac{\det\left[p(\alpha_j - \beta_k)\right]_{j,k=0}^{n-1}}{c^2\Delta(\alpha)\Delta(\beta)} = \frac{1}{(2\pi)^n n!} \int_{h \in \mathbb{R}^n} \Delta(h)^2 \prod_{j=0}^{n-1} e^{-\frac{1}{2}h_j^2} dh_j.$$

Inserting (3.11) and (3.12) into (3.7), we obtain (3.8) after appropriate changes of variables. □

Proposition III.1 follows from Lemma III.5 immediately.

3.3 Proof of Theorem III.2

We apply Theorem II.7 to Proposition III.1. Set

$$(3.13) \qquad d = d_{M,n} := \frac{M\sqrt{n}}{\sqrt{2}\pi},$$

$$(3.14) \qquad \gamma_M(z) = \sin\left(\pi(dz + s)\right)$$

and

(3.15)
$$\mathfrak{C} = \mathfrak{C}_+ \cup \mathfrak{C}_-$$

where the direction of $\mathfrak{C}_+ = \mathbb{R} + i$ with direction from $+\infty + i$ to $-\infty + i$, and $\mathfrak{C}_- = \mathbb{R} - i$ with direction from $-\infty - i$ to $+\infty - i$.

Furthermore, we set

(3.16)
$$\rho(z) = \pm \frac{d}{2}$$

which satisfies

(3.17)
$$\lim_{M \to \infty} \frac{\gamma'_M(z)}{2\pi i \rho(z) \gamma_M(z)} = 1$$

for $z \in \mathfrak{C}_\pm$, as discussed in the paragraph after Theorem II.7. Note that $d^{-n} \mathrm{H}_n(f, \mathcal{D}_{M,s}) = \mathrm{H}_n(d^{-1}f, \mathcal{D}_{M,s})$. Let $p_j(x) = \kappa_j x^j + \cdots$ be the orthonormal polynomials with respect to the weight $f(z)\rho(z)$ on $\mathfrak{C}$, which are exact the same as the orthogonal polynomials with respect to the weight e^{-nx^2} on $\mathbb{R}$ (see Remark II.4). Then from Theorem II.7,

(3.18)
$$\mathbb{P}\left(W_n < M\right) = \int_0^1 P_s(M) ds, \quad P_s(M) = \det\left(1 + K_s\right)_{L^2(\mathfrak{C}_+ \cup \mathfrak{C}_-, dz)},$$

where

(3.19)
$$K_s(z, w) = K_{CD}(z, w) v_s(z)^{\frac{1}{2}} v_s(w)^{\frac{1}{2}} e^{-\frac{n}{2}(z^2 + w^2)}.$$

Here

(3.20)
$$v_s(z) := \begin{cases} -\frac{\cos(\pi(dz+s))}{2i\sin(\pi(dz+s))} - \frac{1}{2} = \frac{e^{2\pi i(dz+s)}}{1 - e^{2\pi i(dz+s)}}, & z \in \mathfrak{C}_+, \\ \frac{\cos(\pi(dz+s))}{2i\sin(\pi(dz+s))} - \frac{1}{2} = \frac{e^{-2\pi i(dz+s)}}{1 - e^{-2\pi i(dz+s)}}, & z \in \mathfrak{C}_-, \end{cases}$$

and K_{CD} is the usual Christoffel-Darboux kernel

(3.21)
$$K_{CD}(z, w) = \frac{\kappa_{n-1}}{\kappa_n} \frac{p_n(z) p_{n-1}(w) - p_{n-1}(z) p_n(w)}{z - w}.$$

We set

$$(3.22) \qquad M = 2\sqrt{n} + 2^{-2/3}n^{-1/6}x.$$

The asymptotic of $P_s(M)$ is obtained in two steps. The first step is to find the asymptotics of the orthonormal polynomials for z in complex plane The second step is to insert them into the formula of K_s and then to prove the convergence of an appropriately scaled operator in trace class. It turns out that the most important information is the asymptotics of the orthonormal polynomials for z close to $z = 0$ with order $n^{-1/3}$. Such asymptotics can be obtained from the method of steepest-descent applied to the integral representation of Hermite polynomials. However, here we proceed using the Riemann-Hilbert method as a way of illustration since the orthonormal polynomials for the other non-intersecting processes to be discussed in the next section are not classical and hence lack the integral representation.

For the weight e^{-nx^2}, the details of the asymptotic analysis of the Riemann-Hilbert problem can be found in [36] and [32]. Let $Y(z)$ be the (unique) 2×2 matrix which (a) is analytic in $\mathbb{C}\backslash\mathbb{R}$, (b) satisfies $Y_+(z) = Y_-(z)\left(\begin{smallmatrix} 1 & e^{-nz^2} \\ 0 & 1 \end{smallmatrix}\right)$ for $z \in \mathbb{R}$, and (c) $Y(z) = (1 + O(z^{-1}))\left(\begin{smallmatrix} z^n & 0 \\ 0 & z^{-n} \end{smallmatrix}\right)$ as $z \to \infty$. It is well-known ([42]) that

$$(3.23) \qquad K_{CD}(z, w) = \frac{Y_{11}(z)Y_{21}(w) - Y_{21}(z)Y_{11}(w)}{-2\pi i(z - w)}.$$

Let

$$(3.24) \qquad g(z) := \frac{1}{\pi} \int_{-\sqrt{2}}^{\sqrt{2}} \log(z - s)\sqrt{2 - s^2}ds$$

be the so-called g-function. Here log denotes the the principal branch of the logarithm. It can be checked that $-g_+(z) - g_-(z) + z^2$ is a constant independent of $z \in (-\sqrt{2}, \sqrt{2})$. Set l to be this constant:

$$(3.25) \qquad l := -g_+(z) - g_-(z) + z^2, \quad z \in (-\sqrt{2}, \sqrt{2}).$$

Set

$$(3.26) \qquad m_\infty(z) := \begin{pmatrix} \frac{\beta+\beta^{-1}}{2} & \frac{\beta-\beta^{-1}}{2i} \\ \frac{\beta-\beta^{-1}}{-2i} & \frac{\beta+\beta^{-1}}{2} \end{pmatrix}, \qquad \beta(z) := \left(\frac{z-\sqrt{2}}{z+\sqrt{2}} \right)^{1/4},$$

where the function $\beta(z)$ is defined to be analytic in $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$ and to satisfy $\beta(z) \to 1$ as $z \to \infty$. Then the asymptotic results from the Riemann-Hilbert analysis is given in Theorem 7.171 in [32]:

$$(3.27) \qquad Y(z) = e^{-\frac{nl}{2}\sigma_3}(I_n + Er(n,z))m_\infty(z)e^{\frac{nl}{2}\sigma_3}e^{ng(z)\sigma_3}, \quad z \in \mathbb{C}\backslash\mathbb{R},$$

where the error term $Er(n,z)$ satisfies (see the remark after theorem 7.171)

$$(3.28) \qquad \sup_{|\operatorname{Im} z|\geq\eta} |Er(n,z)| \leq \frac{C(\eta)}{n}$$

for a positive constant $C(\eta)$, for each $\eta > 0$. An inspection of the proof shows that the same analysis yields the following estimate. The proof is basically the same and we do not repeat.

Lemma III.6. *Let $\eta > 0$. There exists a constant $C(\eta) > 0$ such that for each $0 < \alpha < 1$,*

$$(3.29) \qquad \sup_{z\in D_n} |Er(n,z)| \leq \frac{C(\eta)}{n^{1-\alpha}},$$

where $D_n := \{z : |\operatorname{Im} z| > \frac{\eta}{n^\alpha}, |z \pm \sqrt{2}| > \eta\}$.

We now insert (3.27) into (3.23), and find the asymptotics of K. Before we do so, we first note that the contours $\mathfrak{C}_+$ and $\mathfrak{C}_-$ in the formula of $P_s(M)$ can be deformed thanks to the Cauchy's theorem. We choose the contours as follows, and we call them C_1 and C_2 respectively. Let C_1 be an infinite simple contour in the upper half-plane of shape shown in Figure 3.2 satisfying

$$(3.30) \qquad \operatorname{dist}(\mathbb{R}, C_1) = O(n^{-1/3}), \qquad \operatorname{dist}(\pm\sqrt{2}, C_1) = O(1).$$
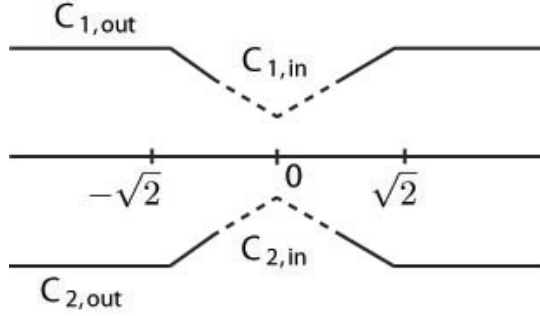
Figure 3.2: $C_1 = C_{1,\text{out}} \cup C_{1,in}, \quad C_2 = C_{2,\text{out}} \cup C_{2,in}$

Set $C_2 = \overline{C_1}$. Later we will make a more specific choice of the contours. Then from Lemma III.6, $Er(n, z) = O(n^{-2/3})$ for $z \in C_1 \cup C_2$. Also since $\beta(z) = O(1)$, $\beta(z)^{-1} = O(1)$, and $\arg(\beta(z)) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ for $z \in C_1 \cup C_2$, we have $\frac{\beta - \beta^{-1}}{\beta + \beta^{-1}} = O(1)$ for $z \in C_1 \cup C_2$. Thus, we find from (3.27) that

$$(3.31) \qquad Y_{11}(z) = e^{ng(z)} \frac{\beta(z) + \beta(z)^{-1}}{2}(1 + O(n^{-2/3}))$$

and

$$(3.32) \qquad Y_{21}(z) = e^{ng(z)+nl}\left(O(n^{-2/3}) + \frac{\beta(z) - \beta(z)^{-1}}{-2i}(1 + O(n^{-2/3}))\right)$$

for $z \in C_1 \cup C_2$. On the other hand, from the definition (3.20) of v_s and the choice of C_1 there exists a positive constant c such that

$$(3.33) \qquad v_s(z) = \begin{cases} e^{2\pi i(dz+s)}(1 + O(e^{-cn^{1/6}})), & z \in C_1, \\ \\ e^{-2\pi i(dz+s)}(1 + O(e^{-cn^{1/6}})), & z \in C_2. \end{cases}$$

Therefore, we find that for $z, w \in C_1 \cup C_2$,

$$(3.34) \qquad K_s(z, w) = \frac{f_1(z)f_2(w) - f_2(z)f_1(w)}{-2\pi i(z - w)} e^{n\phi(z)+n\phi(w)},$$

where

$$(3.35) \qquad \phi(z) := \begin{cases} g(z) - \frac{1}{2}z^2 + \frac{1}{2}l + \frac{iM}{\sqrt{2n}}z, & \mathrm{Im}(z) > 0, \\[2mm] g(z) - \frac{1}{2}z^2 + \frac{1}{2}l - \frac{iM}{\sqrt{2n}}z, & \mathrm{Im}(z) < 0, \end{cases}$$

and f_1, f_2 are both analytic in $\mathbb{C}\backslash\mathbb{R}$ and satisfy

$$(3.36) \qquad f_1(z) = \begin{cases} e^{is\pi \frac{\beta(z)+\beta(z)^{-1}}{2}}(1 + O(n^{-2/3})), & z \in C_1, \\[2mm] e^{-is\pi \frac{\beta(z)+\beta(z)^{-1}}{2}}(1 + O(n^{-2/3})), & z \in C_2, \end{cases}$$

$$(3.37) \qquad f_2(z) = \begin{cases} e^{is\pi}\left(O(n^{-2/3}) + \frac{\beta(z)-\beta(z)^{-1}}{-2i}(1 + O(n^{-2/3})) \right), & z \in C_1, \\[2mm] e^{-is\pi}\left(O(n^{-2/3}) + \frac{\beta(z)-\beta(z)^{-1}}{-2i}(1 + O(n^{-2/3})) \right), & z \in C_2. \end{cases}$$

Note that $f_1(z)$, $f_2(z)$, and their derivatives are bounded on $C_1 \cup C_2$.

So far we only used the fact that the contours C_1 and C_2 satisfy the conditions (3.30). Now we make a more specific choice of the contours as follows (see Figure 3.2). For a small fixed $\epsilon > 0$ to be chosen in Lemma III.7, set

$$(3.38) \qquad \Sigma = \{u + iv : -\epsilon \le u \le \epsilon, v = n^{-1/3} + |u|/\sqrt{3}\}.$$

Define $C_{1,in}$ to be the part of Σ such that $|u| \le n^{-1/4}$:

$$(3.39) \qquad C_{1,in} = \{u + iv : -n^{-1/4} \le u \le n^{-1/4}, v = n^{-1/3} + |u|/\sqrt{3}\}.$$

Define $C_{1,out}$ be the union of $\Sigma \backslash C_{1,in}$ and the horizontal line segments $u + iv_0$, $|u| \ge \epsilon$ where v_0 is the maximal imaginary value of Σ given by $v_0 = n^{-1/3} + \epsilon/\sqrt{2}$. Set $C_1 = C_{1,in} \cup C_{1,out}$. Define $C_2 = \overline{C_1}$. It is clear from the definition that the contours satisfy the conditions (3.30).

Recall that (see (3.22)) $M = 2\sqrt{n} + 2^{-2/3}n^{-1/6}x$ where $x \in \mathbb{R}$ is fixed. We have

Lemma III.7. *There exist $\epsilon > 0$, $n_0 \in \mathbb{N}$, and positive constants c_1 and c_2 such that with the definition (3.38) of Σ with this ϵ, $\phi(z)$ defined in (3.41) satisfies*

(3.40)
$$\mathrm{Re}\ \phi(z) \le c_1 n^{-1}, \qquad z \in C_{1,in} \cup C_{2,in},$$
$$\mathrm{Re}\ \phi(z) \le -c_2 n^{-3/4}, \qquad z \in C_{1,out} \cup C_{2,out},$$

for all $n \ge n_0$.

Proof. From the properties of $g(z)$ and l, it is easy to show that $g(z) - \frac{1}{2}z^2 + \frac{1}{2}l = \int_z^{\sqrt{2}} \sqrt{s^2 - 2}ds$ for $z \in \mathbb{C} \setminus (-\infty, \sqrt{2}]$ (see e.g. (7.60) [32]). Thus,

(3.41)
$$\phi(z) = \int_z^{\sqrt{2}} \sqrt{s^2 - 2}ds \pm \frac{iM}{\sqrt{2n}}z, \qquad z \in \mathbb{C}_\pm.$$

This implies that for $\phi_\pm(u)$ is purely imaginary for $z = u \in (-\sqrt{2}, \sqrt{2})$ where $\phi_\pm$ denotes the boundary values from $\mathbb{C}_\pm$ respectively. Hence for $u \in (-\sqrt{2}, \sqrt{2})$ and $v > 0$, $\mathrm{Re}\ \phi(u + iv) = \mathrm{Re}\ (\phi(u + iv) - \phi_+(u))$. For $u^2 + v^2$ small enough and $v > 0$, using the Taylor series about $s = 0$ and also (3.22), we have

(3.42)
$$\mathrm{Re}\ \phi(u + iv) = -\mathrm{Re}\left(\int_u^{u+iv} \sqrt{s^2 - 2}ds \right) - \frac{Mv}{\sqrt{2n}}$$
$$= -\frac{1}{2^{3/2}}\mathrm{Im}\left(\int_u^{u+iv} (s^2 + O(s^4))ds \right) - \frac{x}{2^{7/6}n^{2/3}}v.$$

The integral involving $O(s^4)$ is $O(|u^2 + v^2|^{5/2})$. On the other hand,

(3.43)
$$-\frac{1}{2^{3/2}}\mathrm{Im}\left(\int_u^{u+iv} s^2 ds \right) - \frac{xv}{2^{7/6}n^{2/3}} = -\frac{1}{2^{2/3}3}(3u^2 v - v^3) - \frac{xv}{2^{7/6}n^{2/3}}.$$

For $z = u + iv$ such that $v = n^{-1/3} + |u|/\sqrt{3}$ (see (3.38)), (3.43) equals

(3.44)
$$n^{-1}\left(-\frac{2^{7/3}}{3^{5/2}}t^3 - \frac{2^{1/3}}{3}t^2 + \frac{(2^{1/2} - x)}{2^{7/6}3^{1/2}}t + \frac{1}{2^{7/6}3}(2^{1/2} - 3x) \right),$$

by setting $t = |u|n^{1/3}$. The polynomial in t is cubic and is of form $f(t) = -a_1 t^3 - a_2 t^2 + a_3 t + a_4$ where $a_1, a_2 > 0$ and $a_3, a_4 \in \mathbb{R}$. It is easy to check that this function is concave down for positive t. Hence

(i) $\sup_{t \geq 0} f(t)$ is bounded above and

(ii) there are $c > 0$ and $t_0 > 0$ such that $f(t) \leq -ct^3$ for $t > t_0$.

Note that for $z \in C_{1,in}$, $t \in [0, n^{1/12}]$. Using (i), we find that (3.44) is bounded above by a constant time n^{-1} for uniformly in $z \in C_{1,in}$. Since the integral involving $O(s^4)$ in (3.42) is $O(n^{-5/4})$ when $z \in C_{1,in}$, we find that there is a constant $c_1 > 0$ such that $\operatorname{Re} \phi(z) \leq c_1 n^{-1}$ for $z \in C_{1,in}$.

Now, for $z = u + iv$ such that $v = n^{-1/3} + |u|/\sqrt{3}$ and $|u| \geq n^{-1/4}$, we have $t = |u|n^{1/3} \geq n^{1/12}$ and hence from (ii), (3.44) is bounded above by $-ct^3 n^{-1} = -c|u|^3$ for all large enough n. On the other hand, for such z, the integral involving $O(s^4)$ in (3.42) is $O(|z|^5) = O(|u|^5)$. Hence $\operatorname{Re} \phi(z) \leq -c|u|^3 + O(|u|^5)$ for such z. Now if we take $\epsilon > 0$ small enough, then there is $c_2 > 0$ such that $\operatorname{Re} \phi(z) \leq -c_2|u|^3$ for $|u| \leq \epsilon$. Combining this, we find that there exist $\epsilon > 0$, $n_0 \in \mathbb{N}$, and $c_2 > 0$ such that for Σ with this ϵ, we have $\operatorname{Re} \phi(z) \leq -c_2|u|^3$ for $z = u + iv \in \Sigma \setminus C_{1,in}$. Since $|u| \geq n^{-1/4}$ for such z, we find $\operatorname{Re} \phi(z) \leq -c_2 n^{-3/4}$ for $z \in \Sigma \setminus C_{1,in}$.

We now fix ϵ as above and consider the horizontal part of $C_{1,out}$. Note that from (3.41), for fixed $v_0 > 0$,

$$(3.45) \qquad \frac{\partial}{\partial u} \operatorname{Re} \phi(u + iv_0) = \operatorname{Re} \phi'(u + iv_0) = -\operatorname{Re} \sqrt{(u + iv_0)^2 - 2}.$$

It is straightforward to check that this is < 0 for $u > 0$ and > 0 for $u < 0$. Hence the value of $\operatorname{Re} \phi(z)$ for z on the horizontal part of $C_{1,out}$ is the largest at the end which are the intersection points of the horizontal segments and Σ. Since $\operatorname{Re} \phi(z) \leq -c_2 n^{-3/4}$ for $z \in \Sigma \setminus C_{1,in}$, we find that the same bound holds for all z on the horizontal segments of $C_{1,out}$. Therefore, we obtain $\operatorname{Re} \phi(z) \leq -c_2 n^{-3/4}$ for all $z \in C_{1,out}$.

The estimates on C_2 follows from the estimates on C_1 due to the symmetry of ϕ about the real axis. $\qquad \square$

Inserting the estimates in Lemma III.7 to the formula (3.34) and using the fact that $f_j(z)$, $j = 1, 2$, and their derivatives are bounded on $C_1 \cup C_2$ (see (3.36) and (3.36)), we find that

$$(3.46) \qquad K_s(z, w) \leq O(e^{-c_2 n^{1/4}}), \qquad \text{if one of } z \text{ or } w \text{ is in } C_{1,\text{out}} \cup C_{2,\text{out}}.$$

We now analyze the kernel $K_s(z, w)$ when $z, w \in C_{1,in} \cup C_{2,in}$. We first scale the kernel. Set

$$(3.47) \qquad \hat{K}_s(\xi, \eta) := 2\pi i \cdot i 2^{1/6} n^{-1/3} K_s(i 2^{1/6} n^{-1/3} \xi, i 2^{1/6} n^{-1/3} \eta).$$

We also set

$$(3.48) \qquad \Sigma_1^{(n)} := \left\{ u + iv : u = 2^{-1/6} + 3^{-1/2}|v|, \; -2^{-1/6} n^{1/12} \leq v \leq 2^{-1/6} n^{1/12} \right\}.$$

This contour is oriented from top to bottom. Note that if $\zeta \in \Sigma_1^{(n)}$, then

$$(3.49) \qquad z = i 2^{1/6} n^{-1/3} \zeta \in C_{1,in}.$$

We also set $\Sigma_2^{(n)} = \{-\xi : \xi \in \Sigma_1^{(n)}\}$ with the orientation from top to bottom. Then

$$(3.50) \qquad \det(1 + K_s)_{L^2(C_{1,in} \cup C_{2,in}, dz)} = \det(1 + \hat{K}_s)_{L^2(\Sigma_1^{(n)} \cup \Sigma_2^{(n)}, \frac{d\zeta}{2\pi i})}.$$

From (3.41),

$$(3.51) \qquad \phi(z) = \begin{cases} \frac{\pi i}{2} + \left(\frac{M - 2\sqrt{n}}{\sqrt{2n}}\right) iz + 2^{-3/2} 3^{-1} i z^3 + O(z^5), & z \in C_{1,in}, \\ -\frac{\pi i}{2} - \left(\frac{M - 2\sqrt{n}}{\sqrt{2n}}\right) iz - 2^{-3/2} 3^{-1} i z^3 + O(z^5), & z \in C_{2,in}. \end{cases}$$

This implies that, using (3.41) and $|z| = O(n^{-1/4})$ for $z \in C_{1,in} \cup C_{2,in}$,

$$(3.52) \qquad n\phi(i 2^{1/6} n^{-1/3} \zeta) = \begin{cases} \frac{n\pi i}{2} + m_x(\zeta) + O(n^{-1/4}), & \zeta \in \Sigma_1^{(n)}, \\ -\frac{n\pi i}{2} - m_x(\zeta) + O(n^{-1/4}), & \zeta \in \Sigma_2^{(n)}, \end{cases}$$

where

$$(3.53) \qquad m_x(\zeta) := -\frac{1}{2}x\zeta + \frac{1}{6}\zeta^3, \qquad \zeta \in \mathbb{C}.$$

It is also easy to check from the definition (3.26) that

$$(3.54) \qquad \beta(i2^{\frac{1}{6}}n^{-\frac{1}{3}}\zeta) = \begin{cases} e^{\frac{i\pi}{4}}\left(1 - i2^{-\frac{4}{3}}n^{-\frac{1}{3}}\zeta + O(n^{-\frac{1}{2}})\right), & \zeta \in \Sigma_1^{(n)}, \\[2mm] e^{\frac{-i\pi}{4}}\left(1 - i2^{-\frac{4}{3}}n^{-\frac{1}{3}}\zeta + O(n^{-\frac{1}{2}})\right), & \zeta \in \Sigma_2^{(n)}. \end{cases}$$

Using these we now evaluate (3.47). Set

$$(3.55) \qquad z = i2^{1/6}n^{-1/3}\xi, \qquad w = i2^{1/6}n^{-1/3}\eta.$$

We consider two cases separately: (a) $z, w \in C_{1,in}$ or $z, w \in C_{2,in}$, and (b) $z \in C_{1,in}, w \in C_{2,in}$, or $z \in C_{2,in}, w \in C_{1,in}$. From (3.54),

$$(3.56) \qquad \frac{\beta(z) - \beta(w)}{z - w} = O(1) \qquad \text{for case (a)},$$

and

$$(3.57) \qquad \frac{\beta(z) - \beta(w)}{z - w} = \pm n^{1/3}\frac{2^{5/6}\sin\frac{\pi}{4}}{\xi - \eta}(1 + O(n^{-\frac{1}{4}})) \quad \text{for case (b)}.$$

Here the sign is $+$ when $z \in C_{1,in}, w \in C_{2,in}$ and $-$ when $z \in C_{2,in}, w \in C_{1,in}$. We also note that using (3.54), for $z \in C_{1,in} \cup C_{2,in}$ the asymptotic formula (3.37) can be expressed as

$$(3.58) \qquad f_2(z) = \begin{cases} e^{is\pi\frac{\beta(z)-\beta(z)^{-1}}{-2i}}\left(1 + O(n^{-5/12})\right), & z \in C_{1,in}, \\[2mm] e^{-is\pi\frac{\beta(z)-\beta(z)^{-1}}{-2i}}\left(1 + O(n^{-5/12})\right), & z \in C_{2,in}. \end{cases}$$

Thus, (3.36), (3.54), and (3.57), implies that for case (b),

$$(3.59)$$
$$\frac{f_1(z)f_2(w) - f_1(z)f_2(w)}{-2\pi i(z - w)} = -\left(\beta(z)^{-1} + \beta(w)^{-1}\right)\frac{\beta(z) - \beta(w)}{4\pi(z - w)}\left(1 + O(n^{-5/12})\right)$$
$$= \mp n^{1/3}\frac{\cos(\frac{\pi}{4})\sin(\frac{\pi}{4})}{2^{1/6}\pi(\xi - \eta)}(1 + O(n^{-\frac{1}{4}})).$$

Inserting this and (3.52) into (3.34) (recall (3.47)), we find that

$$(3.60) \qquad \hat{K}_s(\xi, \eta) = \pm \frac{e^{\pm(m_x(\xi) - m_x(\eta))}}{\xi - \eta}(1 + O(n^{-1/4})),$$

for case (b). A similar calculation using (3.56) instead of (3.57) implies that $\hat{K}_s(\xi, \eta) = O(n^{-1/3})$ for case (a).

The above calculations imply that $\hat{K}_s$ converges to the operator given by the leading term in (3.60) or 0 depending on whether ξ and η are on different limiting contours or on the same limiting contours. From this structure, we find that $\hat{K}_s$ converges to $\begin{pmatrix} 0 & K_{12}^{(\infty)} \\ K_{21}^{(\infty)} & 0 \end{pmatrix}$ on $L^2(\Sigma_1^{(\infty)}, \frac{d\zeta}{2\pi i}) \oplus L^2(\Sigma_2^{(\infty)}, \frac{d\zeta}{2\pi i})$ in the sense of pointwise limit of the kernel where

$$(3.61) \qquad K_{12}^{(\infty)}(\xi, \eta) = \frac{e^{m_x(\xi) - m_x(\eta)}}{\xi - \eta}, \quad K_{21}^{(\infty)}(\xi, \eta) = -\frac{e^{-(m_x(\xi) - m_x(\eta))}}{\xi - \eta},$$

and $\Sigma_1^{(\infty)}$ is a simple contour from $e^{i\pi/3}\infty$ to $e^{-i\pi/3}\infty$ staying in the right half plane, and $\Sigma_2^{(\infty)} = -\Sigma_1^{(\infty)}$ from $e^{2i\pi/3}\infty$ to $e^{-2i\pi/3}\infty$. Note that the limiting kernel does not depend on s.

In order to ensure that the Fredholm determinant also converges to the Fredholm determinant of the limiting operator, we need additional estimates for the derivatives to establish the convergence in trace norm. It is not difficult to check that the formal derivatives of the limiting operators indeed yields the correct limits of the derivatives of the kernel. We do not provide the details of these estimates since the arguments are similar and the calculation follows the standard argument (see [78, Proof of Theorem 3] for an example). Then we obtain

$$(3.62) \qquad \lim_{n \to \infty} \det\left(1 + \hat{K}_s\right)\Big|_{L^2(\Sigma_1^{(n)} \cup \Sigma_2^{(n)}, \frac{d\zeta}{2\pi i})} = \det\left(1 - K_x^{(\infty)}\right)\Big|_{L^2(\Sigma_1^{(\infty)}, \frac{d\zeta}{2\pi i})},$$

where $K_x^{(\infty)} = K_{12}^{(\infty)} K_{21}^{(\infty)}$ of which the kernel is

$$(3.63) \qquad K_x^{(\infty)}(\xi, \eta) := e^{m_x(\xi) + m_x(\eta)} \int_{\Sigma_2^{(\infty)}} \frac{e^{-2m_x(\zeta)}}{(\xi - \zeta)(\eta - \zeta)} \frac{d\zeta}{2\pi i i}.$$

The determinant $\det(1-K_x^{(\infty)})$ equals the Fredholm determinant of the Airy operator. Indeed, this determinant is a conjugated version of the determinant in the paper [78] on ASEP. If we denote the operator in [78, Equation (33)] by $L_s(\eta, \eta')$, then $K_x^{(\infty)}(\xi, \eta) = e^{m_x(\xi)} L_x(\xi, \eta) e^{-m_x(\eta)}$. It was shown in page 153 in [78] that $\det(1+L_s) = F_{GUE}(s)$.

Now, since $\lim_{n\to\infty} P_s(M) = \lim_{n\to\infty} \det(1+K)_{L^2(C_+ \cup C_-)}$ by (3.46), (3.50) and (3.62) implies that $P_s(2\sqrt{n} + 2^{-\frac{2}{3}} n^{-\frac{1}{6}} x) \to F_{GUE}(x)$ for all s. All the estimates are uniform in $s \in [0, 1]$ and we obtain $\mathbb{P}\left(W_n < 2\sqrt{n} + 2^{-2/3} n^{-1/6} x\right) = \int_0^1 P_s(M) ds \to F_{GUE}(x)$. This proves Theorem III.2.

3.4 Continuous-time Symmetric Simple Random Walks

Let $Y(t)$ be a continuous-time symmetric simple random walk, which is defined as follows. The walker initially is at a site of $\mathbb{Z}$. After an exponential waiting time with parameter 1, the walker makes a jump to one of his neighboring site on $\mathbb{Z}$ with equal probability. It is a direct to obtain the transition probability $p_t(x, y) = p_t(y - x)$ where

$$(3.64) \qquad p_t(k) = e^{-t} \sum_{n \in \mathbb{Z}} \frac{(t/2)^{2n+k}}{n!(n+k)!}, \qquad k \in \mathbb{Z}.$$

where $\frac{1}{k!} := 0$ for $k < 0$ by definition.

$Y(t)$ can also be described as the symmetric oscillatory Poisson process [68] or the symmetrized Poisson process [54], which is defined to be the difference of two independent rate $1/2$ Poisson processes. It is easy to see that this difference has the same transition probability as (3.64).

Let $Y_i(t)$ be independent copies of Y and set $X_i(t) = Y_i(t) + i$, $i = 0, 1, 2, \cdots, n-1$. Also set $X(t) := (X_0(t), X_1(t), \cdots, X_{n-1}(t))$. Then $X(0) = (0, 1, \cdots, n-1)$. We condition on the event that (a) $X(T) = X(0)$ and (b) $X_0(t) < X_1(t) < \cdots < X_{n-1}(t)$

for all $t \in [0, T]$. See, for example, [2]. We use the notation $\mathbb{P}$ to denote this conditional probability.

Define the 'width' as

$$(3.65) \qquad W_n(T) = \sup_{t \in [0,T]} (X_{n-1}(t) - X_0(t)).$$

The analogue of Proposition III.1 is the following. The proof is given at the end of this section.

Proposition III.8. *For non-intersecting continuous-time symmetric simple random walks,*

$$(3.66) \quad \mathbb{P}(W_n(T) < M) = \frac{1}{\mathcal{T}_n(f)} \oint_{|s|=1} \mathrm{T}_n(M^{-1}f, \mathcal{D}_{M,s}) \frac{ds}{2\pi i s}, \qquad f(z) = e^{\frac{T}{2}(z+z^{-1})},$$

and $\mathcal{D}_{M,s} = \{z \in \mathbb{C} : z^M = s\}$.

The limit theorem is:

Theorem III.9. *For each $x \in \mathbb{R}$,*

$$(3.67) \qquad \lim_{\min\{n,T\} \to \infty} \mathbb{P}\left(\frac{W_n(T) - \mu(n,T)}{\sigma(n,T)} \le x\right) = F_{GUE}(x)$$

where

$$(3.68) \qquad \mu(n,T) := \begin{cases} 2\sqrt{nT}, & n < T, \\ n + T, & n \ge T, \end{cases}$$

and

$$(3.69) \qquad \sigma(n,T) := \begin{cases} 2^{-2/3}T^{1/3}\left(\sqrt{\frac{n}{T}} + \sqrt{\frac{T}{n}}\right)^{1/3}, & n < T, \\ 2^{-1/3}T^{1/3}, & n \ge T. \end{cases}$$

Note that due to the initial condition and the fact that at most one of X_j's moves with probability 1 at any given time, if X_i is to move downward at time t, it is necessary that $X_0, \cdots, X_{i-1}$ should have moved downward at least once during the time interval $[0, t)$. Thus, if T is small compared to n, then only a few bottom walkers can move downard (and similarly, only a few top walkers can move upward), and hence the middle walkers are 'frozen'(See Figure 3.3). On the other hand, if T is large compared to n, then there is no frozen region. The above result shows that the transition occurs when $T = n$ at which point the scalings (3.68) and (3.69) change.
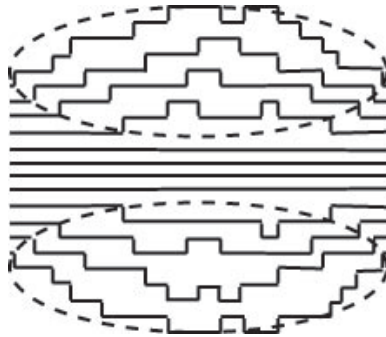

Figure 3.3: Frozen region when $T < n$

Using Theorem II.6, Theorem III.9 can be obtained following the similar analysis as in the subsection 3.3 once we have the asymptotics of the (continuous) orthonormal polynomials with respect to the measure $e^{\frac{T}{2}(z+z^{-1})} \frac{dz}{2\pi i z}$ on the unit circle. The asymptotics of these particular orthonormal polynomials were studied in [9] and [8] using the Deift-Zhou steepest-descent analysis of Riemann-Hilbert problems. In order to be able to control the operator (2.29), the estimates on the error terms in the asymptotics need to be improved. It is not difficult to achieve such estimates by keeping track of the error terms more carefully in the analysis of [9] and [8]. We do not provide any details. Instead we only comment that the difference of the scalings for $n < T$ and $n > T$ is natural from the Riemann-Hilbert analysis of the orthonormal polynomials. If we consider the orthonormal polynomial of degree n, $p_n(z)$, with

weight $e^{\frac{T}{2}(z+z^{-1})}$, the support of the equilibrium measure changes from the full circle when $\frac{n}{T} > 1$ to an arc when $\frac{n}{T} < 1$. The "gap" in the support starts to appear at the point $z = -1$ when $n = T$ and grows as $\frac{n}{T}$ decreases. This results in different asymptotic formulas of the orthonormal polynomials in two different regimes of parameters. However, we point out that the main contribution to the kernel (2.29) turns out to come from the other point on the circle, namely $z = 1$.

For technical reasons, the Riemann-Hilbert analysis is done separately for the following four overlapping regimes of the parameters: (I) $n \geq T + C_1 T^{1/3}$, (II) $T - C_2 T^{1/3} \leq n \leq T + C_3 T^{1/3}$, (III) $c_1 T \leq n \leq T - C_4 T^{1/3}$, (IV) $n \leq c_2 T$ where $0 < c_k < 1$ and $C_k > 0$.

Here we only indicate how the leading order calculation leads to the GUE Tracy-Widom distribution for the case (I). We take

$$(3.70) \qquad M = n + T + 2^{-1/3} T^{1/3} x.$$

Let

$$(3.71) \qquad \mathfrak{C} = \mathfrak{C}_{out} \cup \mathfrak{C}_{in}, \gamma_M(z) = \begin{cases} z^M - s, & z \in \mathfrak{C}_{out}, \\ -z^M + s, & z \in \mathfrak{C}in, \end{cases}$$

where $\mathfrak{C}_{out} = \{z | |z| = 1 + \epsilon\}$ and $\mathfrak{C}_{in} = \{z | |z| = 1 - \epsilon\}$ for some constant $\epsilon > 0$. Set

$$(3.72) \qquad \rho(z) = \begin{cases} M, & z \in \mathfrak{C}_{out}, \\ 0, & z \in \mathfrak{C}_{in}, \end{cases}$$

which satisfies (2.40). Let $p_n(z), \tilde{p}_n(z)$ be the orthonormal polynomial with respect to $M^{-1} f \rho \frac{dz}{2\pi i z}$ on $\mathfrak{C}$ and $\kappa_n, \tilde{\kappa}_n$ be their leading coefficient, see (2.10). Note that f is analytic, these polynomials are exactly the orthogonal polynomials with respect to $f(z) \frac{dz}{2\pi i z}$ on the unit circle Σ. As a result, we have $\tilde{p}_n(z) = p_n(z)$.

By Theorem II.6, we have

$$(3.73) \qquad \frac{T_n(M^{-1}f, \mathcal{D}_{M,s})}{\mathcal{T}_n(f)} = \det(1 + K)$$

where $\det(1 + K)$ is defined in (2.28) with kernel defined in (2.29) with $M^{-1}f$ instead of f.

For case (I), the Riemann-Hilbert analysis implies that

$$(3.74) \qquad \kappa_n^{-1} p_n(z) \approx \begin{cases} z^n e^{-\frac{T}{2}z^{-1}}, & |z| > 1, \\[2mm] o(e^{-\frac{T}{2}z}), & |z| < 1, \end{cases}$$

and

$$(3.75) \qquad \kappa_n p_n^*(z) \approx \begin{cases} o(z^n e^{-\frac{T}{2}z^{-1}}), & |z| > 1, \\[2mm] e^{-\frac{T}{2}z}, & |z| < 1. \end{cases}$$

Here these asymptotics can be made uniform for $|z - 1| \geq O(T^{-1/3})$. In the following, we always assume that z and w satisfy this condition even if we do not state it explicitly. The above estimates imply

$$(3.76)$$

$$(zw)^{-n/2} \frac{p_n(w)p_n^*(z) - p_n(z)p_n^*(w)}{1 - zw^{-1}} \approx \begin{cases} -z^{n/2} \dfrac{e^{-\frac{T}{2}(z^{-1}+w)}}{1 - zw^{-1}} w^{-n/2}, & |z| > 1, |w| < 1, \\[4mm] z^{-n/2} \dfrac{e^{-\frac{T}{2}(z+w^{-1})}}{1 - zw^{-1}} w^{n/2}, & |z| < 1, |w| > 1. \end{cases}$$

The kernel is of smaller order than the above when $|z| < 1, |w| < 1$ or $|z| > 1, |w| > 1$. We also have

$$(3.77) \qquad v(z) := \begin{cases} \dfrac{Ms}{z^M - s} \approx Ms z^{-M}, & |z| > 1, \\[4mm] \dfrac{Mz^M}{s - z^M} \approx \dfrac{M}{s} z^M, & |z| < 1. \end{cases}$$

Here again the approximation is uniform for $|z - 1| \geq O(T^{-1/3})$. Hence inserting $\frac{1}{M}f(z) = \frac{1}{M}e^{\frac{T}{2}(z+z^{-1})}$ to (2.29), we find that the leading order term of the kernel is

$$(3.78) \qquad K(z, w) \approx \pm \frac{e^{\pm(\phi(w)-\phi(z))}}{1 - zw^{-1}}, \qquad \phi(z) := \frac{T}{4}(z - z^{-1}) - \frac{M-n}{2} \log z$$

where the sign is $-$ is when $|z| > 1, |w| < 1$ and is $+$ when $|z| < 1, |w| > 1$. Note that

$$(3.79) \quad \phi(z) = -\frac{T^{1/3}}{2^{4/3}}x(z-1) + \frac{T}{12}(z-1)^3 + O(T^{1/3}(z-1)^2) + O(T(z-1)^4).$$

Hence for $\zeta = O(1)$,

$$(3.80) \qquad \phi(1 + \frac{2^{1/3}}{T^{1/3}}\zeta) = -\frac{1}{2}x\zeta + \frac{1}{6}\zeta^3 + O(T^{-1/3}).$$

After the scaling $z = 1 + \frac{2^{1/3}}{T^{1/3}}\zeta$ and $w = 1 + \frac{2^{1/3}}{T^{1/3}}\eta$, (3.78) converges to the leading term of (3.61), except for the overall sign change which is due to the reverse orientation of the contour. Thus we end up with the same limit (3.62) which is $F_{GUE}(x)$.

3.5 Discrete-time Symmetric Simple Random Walks

Let $X_0(k), \cdots, X_{n-1}(k)$, $k = 0, 1, \cdots, n-1$, be independent discrete-time symmetric simple random walks. Set $X(k) := (X_0(k), X_1(k), \cdots, X_{n-1}(k))$. We take the initial condition as

$$(3.81) \qquad\qquad\qquad X(0) = (0, 2, \cdots, 2n - 2).$$

and consider the process conditional of the event that (a) $X(2T) = X(0)$ and (b) $X_0(k) < X_1(k) < \cdots < X_{n-1}(k)$ for all $k = 0, 1, \cdots, 2T$. The non-intersecting discrete-time simple random walks can also be interpreted as random tiling of a hexagon and were studied in many papers. See, for example, [26, 52, 12, 23]. The notation $\mathbb{P}$ denotes this conditional probability. Define the width

$$(3.82) \qquad\qquad W_n(2T) := \max_{k=0,1,\cdots,2T} \left(X_{n-1}(k) - X_0(k)\right)$$

as before.

Proposition III.10. *For non-intersecting discrete-time symmetric simple random walks,*

(3.83)

$$\mathbb{P}(W_n(2T) < 2M) = \frac{1}{\mathcal{T}_n(f)} \oint_{|s|=1} \mathrm{T}_n(M^{-1}f, \mathcal{D}_{M,s}) \frac{ds}{2\pi i s}, \quad f(z) = z^{-T}(1+z)^{2T},$$

and $\mathcal{D}_{M,s} = \{z \in \mathbb{C} : z^M = s\}$.

The fluctuations are again given by F. Note that $2n \leq W_n(2T) \leq 2n + 2T$ for all n and T.

Theorem III.11. *Fix* $\gamma > 0$ *and* $0 < \beta < 2$. *Then for* $n = [\gamma T^\beta]$,

(3.84)
$$\lim_{T \to \infty} \mathbb{P}\left(\frac{W_n(2T) - 2\sqrt{n^2 + 2nT}}{(n^2 + 2nT)^{-\frac{1}{6}} T^{\frac{2}{3}}} \leq x\right) = F_{GUE}(x).$$

for each $x \in \mathbb{R}$.

Note that the parameter $(n^2 + 2nT)^{-\frac{1}{6}} T^{\frac{2}{3}} \to \infty$ as $T \to \infty$ when $\beta < 2$. This parameter is $O(1)$ when $\beta = 2$. Indeed one can show that when $\beta > 2$,

(3.85)
$$\lim_{T \to \infty} \mathbb{P}(W_n(2T) = 2n + 2T) = 1.$$

The proofs of the proposition and the theorem are similar to those for the continuous-time symmetric simple random walks and we omit them.

CHAPTER IV

Asymptotics of the Ratio of Discrete Toeplitz/Hankel Determinant and its Continuous Counterpart, the Complex Weight Case

4.1   Introduction and Results

In this chapter we consider the case of a discrete Toeplitz determinant $\mathrm{T}_n(f, \mathcal{D})$ where $f(z)$ is a specific complex function (4.4) on $\mathcal{D}$. We compute the asymptotics of the ratio of this discrete Toeplitz determinant and its continuous counterpart $\mathcal{T}_n(f)$. It turns out that one can still apply Theorem II.6 here, but the asymptotic analysis of the Fredholm determinant is much more complicated than that for the cases of real symbols. We develop some new techniques to overcome this complication, which are believed to work for more than this specific complex symbol.

This discrete Toeplitz determinant arises in the width of nonintersecting Poisson process, as described below.

Let $X(t) = (X_0(t), \cdots, X_{n-1}(t))$ be n independent Poisson processes with parameter 1. The transition probability of $X_i(t)(i = 0, 1, \cdots, n-1)$ is given by

$$(4.1) \qquad\qquad p_t(k) = e^{-t}\frac{t^k}{k!}, k = 0, 1, \cdots.$$

Define $D_n := \{x_0 < x_1 < \cdots < x_{n-1}\}$. We condition that (a) $X(0) = (0, 1, \cdots, n-1)$, $X(T) = (a, a+1, \cdots, a+n-1)$ where a is an integer parameter, and (b) $X(t) \in D_n$

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for all $t \in (0, T)$. Denote by $\mathbb{P}$ the conditional probability. Define the width of $X(t)$ $(0 \leq t \leq T)$

$$(4.2) \qquad W_n(T) := \sup_{0 \leq t \leq T} (X_{n-1}(t) - X_0(t)).$$

Similarly to the width of the three nonintersecting processes discussed in the previous chapter, we have the following discrete Toeplitz determinant representation of the width distribution

Proposition IV.1. *For nonintersecting Poisson processes defined above, the distribution of the width is given by*

$$(4.3) \qquad \mathbb{P}(W_n(T) \leq M) = \int_{|s|=r} \frac{\mathrm{T}_n(M^{-1}f, \mathcal{D}_{M,s})}{\mathcal{T}_n(f)} \frac{ds}{2\pi i s}$$

where $r > 0$ is arbitrary, and

$$(4.4) \qquad f(z) := e^{Tz} z^{-a},$$

and

$$(4.5) \qquad \mathcal{D}_{M,s} = \{z | z^M = s^M\}.$$

Remark IV.2. From the formula (4.3), it is easy to check that the distribution of $W_n(T)$ is independent of T.

The main goal of this chapter is to find the asymptotics of (4.3). By using Theorem II.6 and rewriting the kernel of the Fredholm determinant, one can find the following asymptotic result of (4.3).

Theorem IV.3. *Suppose $a = (\gamma + 1)n$ for some constant $\gamma > 0$. Then for any fixed $x \in \mathbb{R}$*

$$(4.6) \qquad \lim_{n \to \infty} \mathbb{P}\left(W_n(T) \leq 2\sqrt{\gamma + 1}n + x\gamma^{2/3}(\gamma + 1)^{-1/6}n^{1/3}\right) = F_{GUE}(x).$$

4.2 Proof of Theorem IV.3

By the Karlin-McGregor argument, it is obvious that $\mathcal{T}_n(f)$ is exactly the probability that n independent Poission processes starting from $X(0) = (0, 1, \cdots, n-1)$ and ending at $X(T) = (a, a+1, \cdots, a+n-1)$ are nonintersecting. Therefore

$$(4.7) \qquad \mathcal{T}_n(f) > 0, n = 1, 2, \cdots,$$

and the orthogonal polynomials $p_k(z) = \gamma_k z^k + \cdots, \tilde{p}_k(z) = \tilde{\gamma}_k z^k$ with respect to the measure $f(z)\frac{dz}{2\pi i z}$

$$(4.8) \qquad \oint_0 p_k(z)\tilde{p}_j(z^{-1})f(z)\frac{dz}{2\pi i z} = \delta_k(j), k, j = 0, 1, \cdots,$$

exist and are unique up to a constant factor.

Now define

$$(4.9) \qquad \gamma_M(z) := z^M - s^M.$$

By using a residue computation, it is a direct to check

$$(4.10) \qquad \int_{\Sigma_{r_+} \cup \Sigma_{r_-}} \frac{\gamma'_M(z)}{2\pi i \gamma_M(z)} z^k M^{-1} f(z) dz = \sum_{z \in \mathcal{D}_{M,s}} z^k M^{-1} f(z),$$

where Σ_{r_+} is a counterclockwise oriented circle centered at the origin and with radius greater than r, and Σ_{r_-} is a clockwise oriented circle centered at the origin and with radius smaller than r. By taking

$$(4.11) \qquad \rho(z) = \begin{cases} M, & |z| > r, \\ 0, & |z| < r, \end{cases}$$

and applying the Theorem II.6, one immediately obtain

Proposition IV.4.

$$(4.12) \qquad \mathrm{T}_n(M^{-1}f, \mathcal{D}_{M,f}) = \mathcal{T}_n(f) \det(I + K_s)$$

where $\det(I + K_s)$ *is a Fredholm determinant defined by*

$$(4.13) \qquad \det(I + K_s) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\Sigma_{r+} \cup \Sigma_{r-})^k} \det(K_s(z_i, z_j))_{i,j=0}^{k-1} \prod_{j=0}^{k-1} \frac{dz_j}{2\pi i z_j}$$

and the kernel

$$(4.14)$$
$$K_s(z, w) = \sqrt{v_s(z) v_s(w) f(z) f(w)} \frac{(z/w)^{\frac{n}{2}} p_n(w) \tilde{p}_n(z^{-1}) - (w/z)^{\frac{n}{2}} p_n(z) \tilde{p}_n(w^{-1})}{1 - z w^{-1}}.$$

Here

$$(4.15) \qquad\qquad v_s(z) := \begin{cases} \frac{z^M}{s^M - z^M}, & |z| < r, \\[2mm] \frac{s^M}{z^M - s^M}, & |z| > r, \end{cases}$$

and $p_j(z), \tilde{p}_j(z)$ *are orthogonal polynomials with respect to* $f(z) \frac{dz}{2\pi i z}$ *on the unit circle*

Σ.

The natural idea is to use the asymptotics of the orthogonal polynomials to find

the asymptotics of the kernel (4.14) and then the asymptotics of the Fredholm de-

terminant (4.13), as what we did in the previous cases. However, it turns out the

idea does not work for this case as explained below.

It is a standard process to find the leading terms of $p_n(z)$ and $\tilde{p}_n(z)$ as n tends

to infinity by using Deift-Zhou steepest descent method. See subsection 4.2.1 for an

illustration (except for the last step of taking $n \to \infty$). One can show the following

$$(4.16) \qquad \log(p_n(z)) = ng(z) + O(1), \log(z^n \tilde{p}_n(z^{-1})) = ng(z) + O(1),$$

where $g(z)$ is the so-called g-function of the corresponding Riemann-Hilbert problem.

Therefore heuristically we have

$$(4.17) \qquad\qquad K_s(z, w) = e^{\frac{n}{2}(\phi(z) + n^{-1} \log v_s(z)) + \frac{n}{2}(\phi(w) + n^{-1} \log v_s(w)) + O(1)},$$

where $\phi(z) = g(z) - \frac{1}{2} \log z + \frac{1}{2} \log f(z)$.

$\phi(z)$ has a property that it has a jump on a specific contour, which we denote by Γ_0. More explicitly, $\Re(\phi)$ reaches its minimum at Γ_0 near the neighborhood of Γ_0. $\phi(z)$ has no other jump. On the other hand, $\Re\log v_s(z)$ has a jump on $\Sigma_r := \{z||z| = r\}$, where it reaches its maximum. In our case when $f(z) = e^{Tz}z^{-a}$, the most important information of the two contours Γ_0 and Σ_r is that they are neither (partly) coincided nor tangent to each other.

Now we consider the asymptotics of $K_s(z,w)$ when $M = 2\sqrt{an} + O(n^{1/3})$ and $a = O(n)$. Then it turns out the main part of $\phi + n^{-1}\log v_s(z)$ has a double critical point z_c. Moreover, z_c is neither on Γ_0 nor on Σ_r. In order to control the kernel, one need to deform both contours Σ_{r+} and Σ_{r-} such that they are close to z_c. However, it is impossible since the kernel has poles on Σ_r which blocks the deformation of Σ_{r+} and Σ_{r-}!

Remark IV.5. In the previous cases we discussed in Chapter II, $f = e^{nV}$ is analytic in $\mathbb{C}\setminus\{0\}$ and is positive on Σ. Under these two conditions, one can show Γ_0, the jump contour of the g-function, is the support of the equilibrium measure $\mu(z)$ on Σ which minimizes the energy function (2.22). Therefore the leading term of the Fredholm kernel K only contains jumps on Σ. On the other hand, the double critical point of this leading term appears exactly on Σ. These two facts are utilized to deform $\mathfrak{C}$ close to the double critical point.

The discussion above tells us that one cannot directly use the asymptotics of the Fredholm kernel K_s to find the asymptotics of the Fredholm determinant (4.13). We need further techniques to solve this problem.

In the next subsection we find a way to rewrite the Fredholm determinant so that the new Fredholm kernel is effective for asymptotic analysis. The idea is to decompose the Christoffel-Darboux kernel part of K_s by using the corresponding
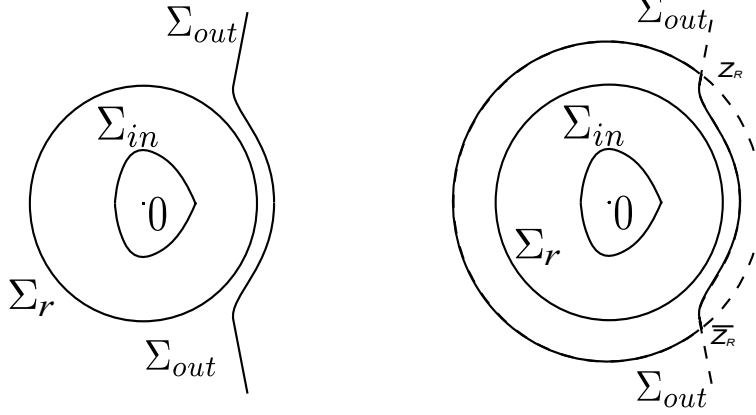
Figure 4.1: The original choice of Σ_{out} and Σ_{in} Figure 4.2: Changing the integral contour Σ_{r+} to Σ_{out}

Riemann-Hilbert problems, and remove the singularities of the v_s part of K_s by a residue computation on the Chiristoffel-Darboux kernel part.

4.2.1 An Identity on the Fredholm Determinant $\det(I + K_s)$

Step 1: Change the Integral Contour

Let Σ_{in} and Σ_{out} be two contours which satisfy the conditions (C1) and (C2) below.

(C1) Σ_{in} is a simple closed contour with clockwise orientation, which encloses the origin and is inside of Σ_r.

(C2) Σ_{out} is a simple and unbounded contour outside of Σ_r which satisfies

$$(4.18) \qquad \Re\left(z + \frac{n - a - M + \lambda}{T} \log z\right) = \text{some negative constant}$$

when $z \in \Sigma_{out}$ becomes sufficiently large. Here $\lambda \geq 0$ is a parameter. The orientation of Σ_{out} is from the lower half plane to the upper half plane. It is easy to see that $\Sigma_{out} \cap \{z \mid |z| > R\}$ is symmetric about the real axis when R tends to infinity by the condition (C2). See Figure 4.1.

Proposition IV.6. *Suppose $\mathfrak{C} = \Sigma_{in} \cup \Sigma_{out}$ satisfies conditions (C1) and (C2). Then*

the Fredholm determinant (4.13) *can be rewritten as*

$$(4.19) \qquad \det(1 + K_s) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathfrak{C}^k} \det(K_s(z_i, z_j))_{i,j=0}^{k-1} \prod_{j=0}^{k-1} \frac{dz_j}{2\pi i z_j}$$

Proof. By the Christoffel-Darboux identity (2.11), it is sufficient to prove

$$(4.20) \qquad \int_{\Sigma_{r+} \cup \Sigma_{r-}} p_i(z) \tilde{p}_i(z^{-1}) f(z) v_s(z) \frac{dz}{2\pi i z} = \int_{\mathfrak{C}} p_i(z) \tilde{p}_i(z^{-1}) f(z) v_s(z) \frac{dz}{2\pi i z}$$

for all $i = 0, 1, \cdots, n-1$, which is equivalent to

$$(4.21) \qquad \int_{\Sigma_{r+} \cup \Sigma_{r-}} z^i f(z) v_s(z) \frac{dz}{2\pi i z} = \int_{\mathfrak{C}} z^i f(z) v_s(z) \frac{dz}{2\pi i z}$$

for all $i = -n, -n+1, \cdots, n-2$. Since Σ_{in} and Σ_{r-} are both simple closed contours enclosing the origin and lying inside of Σ_r, it is sufficient to prove

$$(4.22) \qquad \int_{\Sigma_{r+}} z^i f(z) v_s(z) dz = \int_{\Sigma_{out}} z^i f(z) v_s(z) dz$$

for all $-n \le i \le n-2$.

Set $D_R := \{z | |z| \le R\}$. The condition (C2) of Σ_{out} implies that Σ_{out} intersects with ∂D_R at exact two points if R is large enough, where ∂D_R is the boundary of D_R, i.e., the circle centered at the origin with radius R. Denote the intersection point in the upper half plane by $z_R = e^{i\theta_R}$. Then the second point is $\overline{z_R} = e^{i(2\pi - \theta_R)}$. See Figure 4.2. Therefore

(4.23)

$$\int_{\Sigma_{r+}} z^i f(z) v_s(z) dz = \int_{|z|=R} z^i f(z) v_s(z) dz$$

$$= \int_{|z|=R, \arg z \in (\theta_R, 2\pi - \theta_R)} z^i f(z) v_s(z) dz + \int_{\Sigma_{out} \cap D_R} z^i f(z) v_s(z) dz.$$

Note that the condition (C2) on Σ_{out} implies

$$(4.24) \qquad\qquad\qquad |z^i f(z) v_s(z)| \le |z|^{-2-\lambda}$$

for all $|z| = R$ and $\arg z \in (\theta_R, 2\pi - \theta_R)$. By taking $R \to \infty$ in (4.23), the first term becomes 0 and the second term becomes $\int_{\Sigma_{out}} z^i f(z) v_s(z) dz$. (4.22) follows immediately. $\qquad\square$

Remark IV.7. Notice that K_s is a finite rank operator, and the infinite sum in (4.19) is finite. In fact, $\det(K_s(z_i, z_j))_{i,j=0}^{k-1} = 0$ for $k \geq n$.

Remark IV.8. The definition (4.19) can be interpreted as the Fredholm determinant of an operator on a Hilbert space as follows.

Let $L^2(\mathfrak{C}, |dz|)$ be the Hilbert space of function $h : \mathfrak{C} \to \mathbb{C}$ such that $\int_{\mathfrak{C}} |h(z)|^2 |dz| < \infty$. Now, let $\mathcal{K}_s : L^2(\mathfrak{C}, |dz|) \to L^2(\mathfrak{C}, |dz|)$ be the integral operator with kernel

$$(4.25) \qquad \frac{K_s(z, w)}{2\pi i w} \frac{dw}{|dw|}, z, w \in S.$$

Then

$$(4.26) \qquad (\mathcal{K}_s h)(z) = \int_{\mathfrak{C}} K_s(z, w) h(w) \frac{dw}{2\pi i w}, h \in L^2(\mathfrak{C}, |dz|)$$

and

$$(4.27) \qquad \int_{\mathfrak{C}^k} \det(\mathcal{K}_s(z_i, z_j))_{i,j=0}^{k-1} \prod_{j=0}^{k-1} |dz_j| = \int_{\mathfrak{C}^k} \det(K_s(z_i, z_j))_{i,j=0}^{k-1} \prod_{j=0}^{k-1} \frac{dz_j}{2\pi i z_j}.$$

Therefore (4.19) equals the Fredholm determinant $\det(I + \mathcal{K}_s)$ of the operator $\mathcal{K}_s$ once we show that $\mathcal{K}_s$ is a trace class.

First, we show that $\mathcal{K}_s$ is a bounded operator. Indeed by the choice of Σ_{out},

$$(4.28) \qquad \left| \sqrt{v_s(z) v_s(w) f(z) f(w)} (w/z)^{\frac{n}{2}} p_i(z) \tilde{p}_i(w^{-1}) w^{-1} \right|^2 \leq O(|zw|^{-2-\lambda})$$

for all $0 \leq i \leq n-1$ as $z, w \to \infty$. This implies

$$(4.29) \qquad \int_{\mathfrak{C}} \int_{\mathfrak{C}} \left| \sqrt{v_s(z) v_s(w) f(z) f(w)} (w/z)^{\frac{N}{2}} p_i(z) \tilde{p}_i(w^{-1}) w^{-1} \right|^2 |dw| |dz| < \infty$$

for each i. Therefore

$$(4.30) \qquad \int_{\mathfrak{C}} \int_{\mathfrak{C}} |\mathcal{K}_s(z, w)|^2 |dw||dz| < \infty,$$

which implies that $\mathcal{K}_s$ is bounded and also Hilbert-Schmidt.

Second, since $\mathcal{K}_s$ is a finite rank operator, it is a trace class.

Notation IV.9. For convenience, we say that K_s is a trace class or Hilbert-Schmidt operator in $L^2(\mathfrak{C}, \frac{dz}{2\pi iz})$ if the corresponding operator $\mathcal{K}_s$ defined in (4.26) is trace class or Hilbert-Schmidt in $L^2(\mathfrak{C}, |dz|)$, respectively.

Step 2: Rewrite the Kernel by a Riemann-Hilbert Problem

In order to simplify our notations, we omit the index s in K_s and v_s unless it is necessary.

We suppose all the parameters $N, a(a > N)$ and T are fixed. Therefore the notations $O(1), O(|z|^k)$ are all with respect to the $z \to 0$ or $z \to \infty$ in the complex plane, $(k \in \mathbb{Z})$.

First consider the following Riemann-Hilbert problem.

Riemann-Hilbert Problem IV.10. Find a 2×2 matrix $Y(z)$ satisfying

(a) $Y(z)$ is analytic on $\mathbb{C}\backslash\Sigma$ and is continuous up to the boundary Σ, where Σ denotes the unit circle,

(b) $Y(z) = (I + O(z^{-1}))z^{n\sigma_3}$ as $z \to \infty$,

(c) On Σ, $Y_+(z) = Y_-(z)v_Y(z)$ where

$$(4.31) \qquad v_Y(z) = \begin{pmatrix} 1 & z^{-n}f(z) \\ 0 & 1 \end{pmatrix}.$$

Here $f(z) = e^{Tz}z^{-a}$ is defined in (4.4).

The solution to this Riemann-Hilbert problem exists and is unique. It is given by (see Section 3, [34])

$$(4.32) \qquad Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z), & \int_\Sigma \frac{\kappa_n^{-1} p_n(\xi) f(\xi)}{\xi - z} \frac{d\xi}{2\pi i \xi^n} \\ -\kappa_{n-1} z^{n-1} \tilde{p}_{n-1}(z^{-1}), & \int_\Sigma \frac{-\kappa_{n-1}\tilde{p}_{n-1}(\xi^{-1}) f(\xi)}{\xi - z} \frac{d\xi}{2\pi i \xi} \end{pmatrix}.$$

Therefore we have

$$(4.33) \qquad p_n(z) = \kappa_n Y_{11}(z), \qquad Y_{12}(0) = \kappa_n^{-1} \tilde{\kappa}_n^{-1}.$$

Similarly consider the following Riemann-Hilbert problem

Riemann-Hilbert Problem IV.11. Find a 2×2 matrix $Y(z)$ satisfying

(a) $\tilde{Y}(z)$ is analytic on $\mathbb{C}\backslash\Sigma$ and is continuous up to the boundary Σ, where Σ is the unit circle,

(b) $\tilde{Y}(z) = (I + O(z^{-1}))z^{n\sigma_3}$ as $z \to \infty$,

(c) On Σ, $\tilde{Y}_+(z) = \tilde{Y}_-(z)v_{\tilde{Y}}(z)$ where

$$(4.34) \qquad v_{\tilde{Y}}(z) = \begin{pmatrix} 1 & z^{-n} f(z^{-1}) \\ 0 & 1 \end{pmatrix}.$$

The unique solution is given by

$$(4.35) \qquad \tilde{Y}(z) = \begin{pmatrix} \tilde{\kappa}_n^{-1} \tilde{p}_n(z), & \int_\Sigma \frac{\tilde{\kappa}_n^{-1} \tilde{p}_n(\xi) f(\xi^{-1})}{\xi - z} \frac{d\xi}{2\pi i \xi^n} \\ -\tilde{\kappa}_{n-1} z^{n-1} p_{n-1}(z^{-1}), & \int_\Sigma \frac{-\tilde{\kappa}_{n-1} p_{n-1}(\xi^{-1}) f(\xi^{-1})}{\xi - z} \frac{d\xi}{2\pi i \xi} \end{pmatrix}.$$

Especially we have

$$(4.36) \qquad \tilde{p}_n(z) = \tilde{\kappa}_n \tilde{Y}_{11}(z).$$

Note that the two Riemann-Hilbert problems are related. It is easy to check that they satisfy the following relation

$$(4.37) \qquad \tilde{Y}(z) = Y(0)^{-1} Y(z^{-1}) z^{n\sigma_3}.$$

Hence we have

$$(4.38) \qquad \tilde{Y}_{11}(z) = (Y_{22}(0)Y_{11}(z^{-1}) - Y_{12}(0)Y_{21}(z^{-1}))z^n.$$

By plugging in (4.33) and (4.36) into the kernel K_s (4.14), we have

(4.39)
$$K_s(z,w) = \sqrt{v_s(z)v_s(w)f(z)f(w)} \frac{(z/w)^{n/2}Y_{11}(w)\tilde{Y}_{11}(z^{-1}) - (w/z)^{n/2}Y_{11}(z)\tilde{Y}_{11}(w^{-1})}{Y_{12}(0)(1 - zw^{-1})}.$$

Then we plug in (4.38) and obtain

Proposition IV.12. *Let $Y(z)$ be the unique solution to the Riemann-Hilbert problem IV.10, then*

$$(4.40) \qquad K_s(z,w) = \sqrt{v_s(z)v_s(w)f(z)f(w)}(zw)^{-n/2}\frac{Y_{11}(z)Y_{21}(w) - Y_{11}(w)Y_{21}(z)}{1 - zw^{-1}}.$$

Remark IV.13. If we write $\mathrm{T}_n(M^{-1}f, \mathcal{D}_{M,s})$ and $\mathcal{T}_n(f)$ as the corresponded Hankel determinants $(-1)^{(n-1)(n-2)/2}\mathrm{H}_n(M^{-1}z^{-n+1}f, \mathcal{D}_{M,s})$ and $(-1)^{(n-1)(n-2)/2}\mathcal{H}_n(z^{-n+1}f)$, and apply Theorem II.7, we can arrive at the same Fredholm determinant with the kernel (4.40).

Step 3: Deform the Riemann-Hilbert Problem and Further Rewrite the Fredholm Determinant

Let

$$(4.41) \qquad \xi := \frac{a - n + 2i\sqrt{an}}{T},$$

and Γ be a simple contour from $\bar{\xi}$ to ξ which will be defined explicitly later. Here we just require Γ does not intersect $(-\infty, 0]$.

Define the g-function of the Riemann-Hilbert problem IV.10 as following

$$(4.42) \qquad g(z) := \int_{\xi}^{z} \left(-\frac{T}{2n} + \frac{a+n}{2n}\frac{1}{z} + \frac{T}{2n}\frac{\sqrt{(z-\xi)(z-\bar{\xi})}}{z} \right) dz - l',$$

where $\sqrt{(z-\xi)(z-\bar\xi)}$ is defined in such a way that it has the branch cut Γ and behaves like z as $z \to \infty$, and l' is a constant determined later. We also require the integral contour does not intersect Γ. Note that the integrand does not have a pole at $z = 0$ due to the choice of ξ in (4.41). It is easy to check that $g(z)$ is a multi-valued "function" varying by multiples of $2\pi i$, depending on the number of times the integral curve travels around Γ. Therefore $e^{ng(z)}$ is well-defined as a single-valued function. Note that $g'(z) - \frac{1}{z} = O(|z|^{-2})$ as $z \to \infty$. Hence we have $e^{ng(z)}z^{-n} = O(z^{-1})$ as $z \to \infty$.

Now we do the standard deformations (see [32] for example). Notice that the choice of the jump contour in the Riemann-Hilbert problem IV.10 does not affect the first column of $Y(z)$, as long as it is a simple closed contour enclosing 0. Therefore we can pick $\Gamma \cup \Gamma'$ as the jump contour, where Γ' is a simple curve from ξ to $\bar\xi$ such that Γ' does not intersect $[0, +\infty)$. We will put other restrictions on Γ' later. Define

$$(4.43) \qquad Q(z) := e^{nl\sigma_3/2}Y(z)e^{-ng(z)\sigma_3}e^{-nl\sigma_3/2},$$

where l is the constant satisfying

$$(4.44) \qquad ng(z)_+ + ng_-(z) + Tz - (a+n)\log z + nl = 0,$$

for all $z \in \Gamma$.

Then $Q(z)$ solves the following

Riemann-Hilbert Problem IV.14. Find a 2×2 matrix $Q(z)$ satisfying

(a) $Q(z)$ is analytic except on $\mathbb{C}\backslash(\Gamma \cup \Gamma')$ and is continuous up to the boundary $\Gamma \cup \Gamma'$,

(b) $Q(z) = I + O(z^{-1})$ as $z \to \infty$,

(C) On the jump contour, $Q_+(z) = Q_-(z)v_Q(z)$ where

$$(4.45) \qquad v_Q(z) = \begin{cases} \begin{pmatrix} e^{2n\phi_-(z)} & 1 \\ 0 & e^{2n\phi_+(z)} \end{pmatrix}, & z \in \Gamma, \\[2em] \begin{pmatrix} 1 & e^{2n\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma', \end{cases}$$

where

$$(4.46) \qquad \phi(z) := \int_\xi^z \frac{T}{2n} \frac{\sqrt{(z-\xi)(z-\bar\xi)}}{z} dz = g(z) + \frac{T}{2n} z - \frac{n+a}{2n} \log z + \frac{l}{2}$$

with Γ as its branch cut. Here the integral contour cannot pass 0 or intersect Γ.

$\phi(z)$ is a multi-valued "function" varying by multiples of $2\pi i$ and

$$(4.47) \qquad \oint_0 \frac{T}{2n} \frac{\sqrt{(z-\xi)(z-\bar\xi)}}{z} dz = \frac{(-a-n)\pi i}{n},$$

depending on the number of times the integral contour travels around Γ and the origin. Therefore $e^{2n\phi}$ is well defined as a single-valued function, and $\Re(\phi(z))$ is independent of the integral contour.

By symmetry we have

$$\begin{aligned}
(4.48) \qquad \Re(\phi(\bar\xi)) &= \Re\left(\int_\xi^{\bar\xi} \frac{T}{2n} \frac{\sqrt{(z-\xi)(z-\bar\xi)}}{z} dz \right) \\
&= \frac{T}{4n} \int_\xi^{\bar\xi} \left(\frac{\sqrt{(z-\xi)(z-\bar\xi)}}{z} dz + \frac{\sqrt{(\bar z-\xi)(\bar z-\bar\xi)}}{\bar z} d\bar z \right) \\
&= \frac{T}{4n} \int_\xi^{\bar\xi} \frac{\sqrt{(z-\xi)(z-\bar\xi)}}{z} dz + \frac{T}{4n} \int_{\bar\xi}^\xi \frac{\sqrt{(z-\xi)(z-\bar\xi)}}{z} dz \\
&= 0,
\end{aligned}$$

where the integral contour is chosen to be symmetric about the real axis and does not intersect Γ.

Furthermore, for any $z \in \mathbb{C}$, we have

$$
\Re(\phi(z)) = \Re\left(\int_\xi^z \frac{T}{2n} \frac{\sqrt{(z-\xi)(z-\bar{\xi})}}{z} dz\right)
$$

$$
= \frac{T}{4n} \int_\xi^z \left(\frac{\sqrt{(z-\xi)(z-\bar{\xi})}}{z} dz + \frac{\sqrt{(\bar{z}-\xi)(\bar{z}-\bar{\xi})}}{\bar{z}} d\bar{z}\right)
$$

(4.49)
$$
= \frac{T}{4n} \int_{\bar{\xi}}^{\bar{z}} \left(\frac{\sqrt{(\bar{z}-\xi)(\bar{z}-\bar{\xi})}}{\bar{z}} d\bar{z} + \frac{\sqrt{(z-\xi)(z-\bar{\xi})}}{z} dz\right)
$$

$$
= \Re\left(\int_{\bar{\xi}}^{\bar{z}} \frac{T}{2n} \frac{\sqrt{(z-\xi)(z-\bar{\xi})}}{z} dz\right)
$$

$$
= \Re(\phi(\bar{z})).
$$

Therefore $\Re(\phi(z))$ is symmetric about the real axis.

By expanding $\phi'(z)$ at ∞, it is easy to see

(4.50)
$$
\phi(z) = \frac{T}{2n} z - \frac{a-n}{2n} \log z + O(1)
$$

as $z \to \infty$. Also note that $\phi(z) = -\frac{T|\xi|}{2n} \log(z) + O(1)$ near 0, $\phi(z) = c(z-\xi)^{3/2} + O(z-\xi)^{5/2}$ near ξ, where c is a nonzero constant which can be computed explicitly. By a standard topological discussion on the quadratic differentials (see [63, 10] for examples) one can show the set $\{z | \Re(\phi(z)) = 0\}$ is independent of the choice of Γ. It consists of four segments with endpoints ξ or/and $\bar{\xi}$, as shown in Figure 4.3. The two bounded segments form a region which contains 0. Denote by Γ_0 the finite segment to the right of the origin, Γ_1 the finite segment to the left of the origin, and Γ_2 the union of the other two infinite segments. Note that $\phi'(z)$ is nonzero on the real axis, both Γ_0 and Γ_1 intersect $\mathbb{R}$ at exact one point.

Now we choose $\Gamma = \Gamma_0$. Denote by $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ the regions from left to right which are divided by Γ, Γ_1 and Γ_2. See Figure 4.3. Then it is easy to see $\Re(\phi(z)) < 0$ for all $z \in \mathcal{O}_1$, and $\Re(\phi(z)) > 0$ for all $z \in \mathcal{O}_2 \cup \mathcal{O}_3$.

We have more discussions about the properties of Γ, Γ_1 and Γ_2 later.
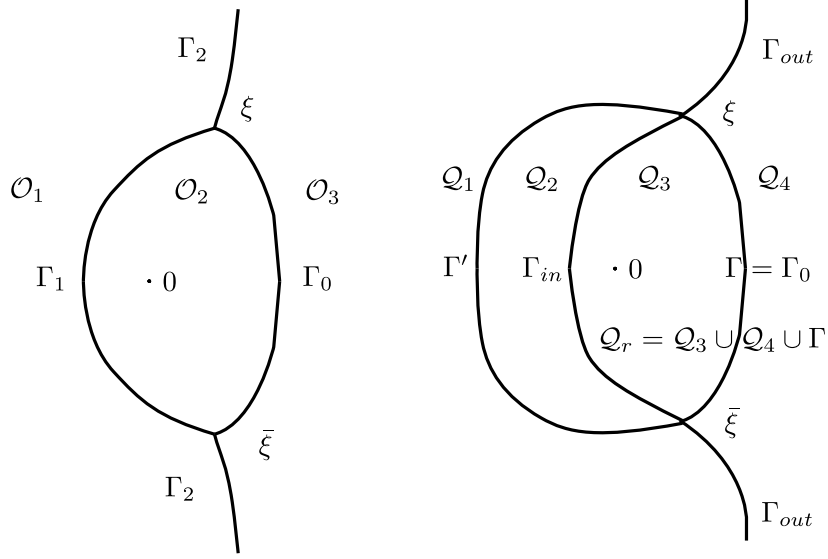
Figure 4.3: The graph of $\Gamma \cup \Gamma_1 \cup \Gamma_2 = \{z | \Re(\phi(z)) = 0\}$

Figure 4.4: The jump contours of the Riemann-Hilbert problem IV.15

Now come back to the Riemann-Hilbert problem IV.14. Choose Γ' to be an open contour from ξ to $\bar{\xi}$ and which lies in $\mathcal{O}_1$. Then $\Re(\phi(z)) < 0$ for all $z \in \Gamma'$.

Up to now all the deformations are standard. For the next step, instead of opening two lenses close to Γ_0 as usual, we use certain specific unbounded lenses going to infinity. More explicitly, define three contours as following: Γ_{in} is a contour from $\bar{\xi}$ to $\bar{\xi}$ which lies in $\mathcal{O}_2$. We also require that 0 is to the right of Γ_{in}. The upper part of Γ_{out} is a contour from ∞ to ξ which lies in $\mathcal{O}_3$ and also in the upper half plane. Moreover, when z becomes large along Γ_{out},

$$(4.51) \qquad \Re\left(\phi(z) - \frac{M}{4n}\log z\right) = 0.$$

This contour exists since that $\Re\left(\phi(z) - \frac{M}{4n}\log z\right)$ is a monic function of $\Re(z)$ in any large enough circle $|z| = R$, and that $\Re\log(z) > 0$ for all $|z| > 1$. We complete Γ_{out} by making it symmetric about the real axis. See Figure 4.4.

The contours $\Gamma', \Gamma_{in}, \Gamma,$ and Γ_{out} divide the complex plane into four regions. We denote by $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3,$ and $\mathcal{Q}_4$ the four regions from left to right. Note that our choice

of Γ_{in} implies $0 \in \mathcal{Q}_3$. We also denote $\mathcal{Q}_r$ the region which is bounded by $\Gamma_{in} \cup \Gamma_{out}$ and contains $\mathbb{R}_+$. Then $\mathcal{Q}_r = \mathcal{Q}_3 \cup \mathcal{Q}_4 \cup \Gamma$.

We will show $\mathcal{Q}_r$ has a nice property that the solution matrix to the final Riemann-Hilbert problem is analytic in $\mathcal{Q}_r$, see the Riemann-Hilbert problem IV.16. This property allows us to compute the residues and cancel terms in the expansion of Fredholm determinant $\det(1 + K)$ if Σ_{in} and Σ_{out} are chosen in $\mathcal{Q}_r$ (see Proposition IV.17). Recall that Σ_{out} satisfies (4.18), which implies

$$(4.52) \qquad \Re\left(\phi(z) - \frac{M - \lambda}{2n} \log z\right) = \text{constant}$$

as $z \to \infty$ along Σ_{out}. Our choice of Γ_{out} make it possible to choose a contour Σ_{out} in $\mathcal{Q}_r$ if $\lambda < \frac{M}{2}$. It is also possible to choose Σ_{in} in $\mathcal{Q}_r$ since $0 \in \mathcal{Q}_3 \subset \mathcal{Q}_r$. See Figure 4.7 and its interpretation later.

Now we define

$$(4.53) \qquad S(z) := \begin{cases} Q(z), & z \in \mathcal{Q}_1 \cup \mathcal{Q}_2, \\[2em] Q(z) \begin{pmatrix} 1 & 0 \\ -e^{-2n\phi(z)} & 1 \end{pmatrix}, & z \in \mathcal{Q}_3, \\[2em] Q(z) \begin{pmatrix} 1 & 0 \\ e^{-2n\phi(z)} & 1 \end{pmatrix}, & z \in \mathcal{Q}_4. \end{cases}$$

Here, although $\phi(z)$ has a pole at $z = 0$, $e^{-2n\phi(z)}$ is analytic at $z = 0$ by (4.46). Therefore S is analytic in $\mathcal{Q}_3$.

$S(z)$ solves the following Riemann-Hilbert problem:

Riemann-Hilbert Problem IV.15. Find a 2×2 matrix $S(z)$ satisfying

(a) $S(z)$ is analytic on $\mathbb{C}\backslash(\Gamma \cup \Gamma' \cup \Gamma_{in} \cup \Gamma_{out})$, and is continuous up to the boundaries $\Gamma \cup \Gamma' \cup \Gamma_{in} \cup \Gamma_{out}$,

(b) $S(z) = I + O(z^{-1})$ as $z \to \infty$,

(C) On the jump contours, $S_+(z) = S_-(z)v_S(z)$ where

$$
(4.54) \qquad v_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-2n\phi(z)} & 1 \end{pmatrix}, & z \in \Gamma_{in} \cup \Gamma_{out}, \\[2em] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Gamma, \\[2em] \begin{pmatrix} 1 & e^{2n\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma'. \end{cases}
$$

Define

$$
(4.55) \qquad R(z) := S(z) \begin{pmatrix} \frac{\beta + \beta^{-1}}{2} & \frac{\beta - \beta^{-1}}{2i} \\ \frac{\beta - \beta^{-1}}{-2i} & \frac{\beta + \beta^{-1}}{2} \end{pmatrix}^{-1}
$$

where $\beta(z) = \left(\frac{z-\xi}{z-\tilde{\xi}} \right)^{1/4}$ which has the branch cut Γ and $\beta(z) = 1 + O(z^{-1})$ as $z \to \infty$.

Then $R(z)$ satisfies the following Riemann-Hilbert problem

Riemann-Hilbert Problem IV.16. Find a 2×2 matrix $R(z)$ satisfying

(a) $R(z)$ is analytic on $\mathbb{C} \backslash (\Gamma' \cup \Gamma_{in} \cup \Gamma_{out})$ and is continuous up to the boundaries $\Gamma' \cup \Gamma_{in} \cup \Gamma_{out}$,

(b) $R(z) = I + O(z^{-1})$ as $z \to \infty$,

(C) On the jump contours, $R_+(z) = R_-(z)v_R(z)$ where

$$
(4.56) \qquad v_R(z) = \begin{cases} I + e^{-2n\phi(z)} E(z), & \Gamma_{in} \cup \Gamma_{out}, \\[1em] I + e^{2n\phi(z)} E(z), & z \in \Gamma'. \end{cases}
$$

Here $E(z)$ is defined by

$$
(4.57) \qquad E(z) := \begin{cases} \begin{pmatrix} \frac{\beta-\beta^{-1}}{2i}\frac{\beta+\beta^{-1}}{2} & -\frac{\beta-\beta^{-1}}{2i}\frac{\beta-\beta^{-1}}{2i} \\[6pt] \frac{\beta+\beta^{-1}}{2}\frac{\beta+\beta^{-1}}{2} & \frac{\beta+\beta^{-1}}{2}\frac{\beta-\beta^{-1}}{-2i} \end{pmatrix}, & z \in \Gamma_{in} \cup \Gamma_{out}, \\[20pt] \begin{pmatrix} -\frac{\beta+\beta^{-1}}{2}\frac{\beta-\beta^{-1}}{-2i} & \frac{\beta+\beta^{-1}}{2}\frac{\beta+\beta^{-1}}{2} \\[6pt] -\frac{\beta-\beta^{-1}}{-2i}\frac{\beta-\beta^{-1}}{-2i} & \frac{\beta-\beta^{-1}}{-2i}\frac{\beta+\beta^{-1}}{2} \end{pmatrix}, & z \in \Gamma'. \end{cases}
$$

Note that $R(z)$ is analytic in the region $\mathcal{Q}_r$. Moreover, by using the fact that $\det(Y(z)) = 1$ for all $z \in \mathbb{C}$ since $\det(Y(z))$ is an entire function and goes to 1 at ∞, and the fact that our deformations do not change the determinant of the desired matrices, we have $\det(R(z)) = \det(S(z)) = \det(Q(z)) = \det(Y(z)) = 1$ for all $z \in \mathbb{C}$. Hence we have

$$
(4.58) \qquad R_{11}(z)R_{22}(z) - R_{12}(z)R_{21}(z) = 1
$$

for all $z \in \mathbb{C}\backslash(\Gamma' \cup \Gamma_{in} \cup \Gamma_{out})$. $R(z)$ also has simple asymptotic behaviors as the parameters go to infinity simultaneously: $R(z) = I + O(n^{-1})$ for any fixed z in $\mathbb{C}\backslash(\Gamma' \cup \Gamma_{in} \cup \Gamma_{out})$. See the next subsection. We do not use this property in this subsection since we suppose all the parameters are fixed.

Now we rewrite the Fredholm kernel (4.40) in terms of $R(z)$. Assume the following condition on Σ_{in} and Σ_{out} holds.

(C3) Σ_{in} and Σ_{out} are both in $\mathcal{Q}_r$.

By using (4.46), we have

$$
(4.59) \qquad K(z,w) = \sqrt{v(z)v(w)}\frac{F(z)G(w) - F(w)G(z)}{1 - zw^{-1}},
$$

where $F(z) := e^{n\phi(z)}Y_{11}(z)e^{-ng(z)}$ and $G(z) := e^{n\phi(z)}Y_{21}(z)e^{-ng(z)-nl}$ for all z in $\mathbb{C}\backslash(\Gamma \cup$

Γ'). By tracking the relation between $Y(z)$ and $R(z)$ we have

$$(4.60) \qquad F(z) = \begin{cases} e^{n\phi(z)}\left(R_{11}(z)\frac{\beta+\beta^{-1}}{2} + R_{12}(z)\frac{\beta-\beta^{-1}}{-2i}\right) \\ \quad + e^{-n\phi(z)}\left(R_{11}(z)\frac{\beta-\beta^{-1}}{2i} + R_{12}(z)\frac{\beta+\beta^{-1}}{2}\right), \quad z \in \mathcal{Q}_3, \\ e^{n\phi(z)}\left(R_{11}(z)\frac{\beta+\beta^{-1}}{2} + R_{12}(z)\frac{\beta-\beta^{-1}}{-2i}\right) \\ \quad - e^{-n\phi(z)}\left(R_{11}(z)\frac{\beta-\beta^{-1}}{2i} + R_{12}(z)\frac{\beta+\beta^{-1}}{2}\right), \quad z \in \mathcal{Q}_4, \end{cases}$$

and

$$(4.61) \qquad G(z) = \begin{cases} e^{n\phi(z)}\left(R_{21}(z)\frac{\beta+\beta^{-1}}{2} + R_{22}(z)\frac{\beta-\beta^{-1}}{-2i}\right) \\ \quad + e^{-n\phi(z)}\left(R_{21}(z)\frac{\beta-\beta^{-1}}{2i} + R_{22}(z)\frac{\beta+\beta^{-1}}{2}\right), \quad z \in \mathcal{Q}_3, \\ e^{n\phi(z)}\left(R_{21}(z)\frac{\beta+\beta^{-1}}{2} + R_{22}(z)\frac{\beta-\beta^{-1}}{-2i}\right) \\ \quad - e^{-n\phi(z)}\left(R_{21}(z)\frac{\beta-\beta^{-1}}{2i} + R_{22}(z)\frac{\beta+\beta^{-1}}{2}\right), \quad z \in \mathcal{Q}_4. \end{cases}$$

Here we omit the case $z \in \mathcal{Q}_1 \cup \mathcal{Q}_2$ since $\Sigma_{in} \cup \Sigma_{out}$ lies in $\mathcal{Q}_r = \mathcal{Q}_3 \cup \mathcal{Q}_4 \cup \Gamma$.

It is easy to check (from the definitions or the formulas above) that $F(z)$ and $G(z)$ are both analytic in $\mathcal{Q}_r \backslash \{0\}$. Each of them can be decomposed to the sum of two functions which are also analytic in $\mathcal{Q}_r \backslash \{0\}$. More precisely, $F(z) = F_0(z) + F_1(z)$ and $G(z) = G_0(z) + G_1(z)$, where $F_0(z), G_0(z)$ are analytic functions in $\mathcal{Q}_r \backslash \{0\}$ defined by

$$(4.62) \qquad F_0(z) = \begin{cases} e^{-n\phi(z)}\left(R_{11}(z)\frac{\beta-\beta^{-1}}{2i} + R_{12}(z)\frac{\beta+\beta^{-1}}{2}\right), \quad z \in \mathcal{Q}_3, \\ e^{n\phi(z)}\left(R_{11}(z)\frac{\beta+\beta^{-1}}{2} + R_{12}(z)\frac{\beta-\beta^{-1}}{-2i}\right), \quad z \in \mathcal{Q}_4, \end{cases}$$

$$(4.63) \qquad G_0(z) = \begin{cases} e^{-n\phi(z)}\left(R_{21}(z)\frac{\beta-\beta^{-1}}{2i} + R_{22}(z)\frac{\beta+\beta^{-1}}{2}\right), \quad z \in \mathcal{Q}_3, \\ e^{n\phi(z)}\left(R_{21}(z)\frac{\beta+\beta^{-1}}{2} + R_{22}(z)\frac{\beta-\beta^{-1}}{-2i}\right), \quad z \in \mathcal{Q}_4, \end{cases}$$

and $F_1(z)$, $G_1(z)$ are analytic functions in $\mathcal{Q}_r\backslash\{0\}$ defined by

$$(4.64) \qquad F_1(z) = \begin{cases} e^{n\phi(z)}\left(R_{11}(z)\frac{\beta+\beta^{-1}}{2} + R_{12}(z)\frac{\beta-\beta^{-1}}{-2i}\right), & z \in \mathcal{Q}_3, \\[2mm] -e^{-n\phi(z)}\left(R_{11}(z)\frac{\beta-\beta^{-1}}{2i} + R_{12}(z)\frac{\beta+\beta^{-1}}{2}\right), & z \in \mathcal{Q}_4, \end{cases}$$

$$(4.65) \qquad G_1(z) = \begin{cases} e^{n\phi(z)}\left(R_{21}(z)\frac{\beta+\beta^{-1}}{2} + R_{22}(z)\frac{\beta-\beta^{-1}}{-2i}\right), & z \in \mathcal{Q}_3, \\[2mm] -e^{-n\phi(z)}\left(R_{21}(z)\frac{\beta-\beta^{-1}}{2i} + R_{22}(z)\frac{\beta+\beta^{-1}}{2}\right), & z \in \mathcal{Q}_4. \end{cases}$$

There are three properties of $F_i(z), G_i(z)$ $(i = 0, 1)$ which turn out to be the key to simplify the Fredholm determinant $\det(1 + K)$. Suppose Σ_{in} is a simple closed contour in $\mathcal{Q}_r$ which encloses 0, and Σ_{out} is a simple contour in $\mathcal{Q}_r$ which separates 0 and $+\infty$. We also assume $\int_{\Sigma_{out}} |z|^{-2}|dz| < \infty$ which implies that Σ_{out} tends to ∞ nicely. Note that this assumption is satisfied when (4.18) holds for large enough z on Σ_{out}. Denote by $\mathcal{P}_0$ the region bounded by Σ_{in}, and denote by $\mathcal{P}_1$ the region to the right of Σ_{out}. Then $(\mathcal{P}_0 \cup \mathcal{P}_1) \subset \mathcal{Q}_r$.

Property 1 (P1): For any function $h_0(z)$ which is analytic in $\mathcal{P}_0$ and continuous to the boundary Σ_{in}, we have

$$(4.66) \qquad \begin{aligned} \int_{\Sigma_{in}} h_0(z)F_0(z)F_i(z)dz = 0, \quad \int_{\Sigma_{in}} h_0(z)G_0(z)G_i(z)dz = 0, \\ \int_{\Sigma_{in}} h_0(z)F_0(z)G_i(z)dz = 0, \quad \int_{\Sigma_{in}} h_0(z)G_0(z)F_i(z)dz = 0, \end{aligned}$$

for all $i = 0, 1$.

Property 2 (P2): Suppose $h_1(z)$ is an analytic function in $\mathcal{P}_1$ which is continuous to the boundary Σ_{out}. Moreover, $h_1(z) = O(|z|^{-2})$ as $z \to \infty$ in $\mathcal{P}_1$. Then we have

$$(4.67) \qquad \begin{aligned} \int_{\Sigma_{out}} h_1(z)F_1(z)F_i(z)dz = 0, \quad \int_{\Sigma_{out}} h_1(z)G_1(z)G_i(z)dz = 0, \\ \int_{\Sigma_{out}} h_1(z)F_1(z)G_i(z)dz = 0, \quad \int_{\Sigma_{out}} h_1(z)G_1(z)F_i(z)dz = 0, \end{aligned}$$

for all $i = 0, 1$.

Property 3 (P3): For any $z \in \mathcal{Q}_r$, we have

$$(4.68) \qquad\qquad F_1(z)G_0(z) - F_0(z)G_1(z) = 1.$$

Now we show the proof of these properties.

By the discussions after (4.53) we know $e^{-2n\phi(z)}$ is analytic in $\mathcal{P}_0$. Together with the analyticity of $\beta(z)$ and $R(z)$ in $\mathcal{Q}_r$, we obtain that the integrands of (4.66) are analytic in $\mathcal{P}_0$. Therefore (P1) holds.

Note that $0 \notin \mathcal{P}_1$. So the integrands of (4.67) are analytic in $\mathcal{P}_1$. We change the integral contour Σ_{out} to another contour which is the union of $\Sigma_{out} \cap \{z \,|\, |z| > C\}$ and an arc in $\{z \,|\, |z| = C\} \cap \mathcal{Q}_r$, where C is a large constant. Since $\Re(\phi(z)) > 0$, $R(z) \to I$, $\beta(z) \to 1$, and $h_1(z) = O(|z|^{-2})$ as $z \to \infty$ in $\mathcal{Q}_r$, we have that the integral on the arc is of order $O(C^{-1})$. Also note that the assumption on Σ_{out} implies the integral on $\Sigma_{out} \cap \{z \,|\, |z| > C\}$ tends to 0 as C goes to inifinity. Therefore (P2) follows by taking $C \to \infty$.

It is direct to check the last property by using (4.58).

Recall we have the Fredholm kernel (4.59). The property (P1) hints that the $F_0(z), G_0(z)$ parts might be canceled in the expansion of the Fredholm determinant when $z \in \Sigma_{in}$. Similarly, the property (P2) hints that $F_1(z), G_1(z)$ parts might be canceled when $z \in \Sigma_{out}$. The property (P3) implies that we can compute the residue of the kernel at $z = w$. Therefore it helps us to evaluate the integrals even though the integrand has a pole at $z = w$.

Now we state the main result of this section.

Proposition IV.17. *Suppose $M \geq 2$ is a positive integer, and suppose Σ_{in} and Σ_{out} are two contours which satisfy the conditions (C1), (C2) with parameter $1 < \lambda < 2$*

and (C3). We have

$$(4.69) \qquad \det(1 + K) = \det(1 + \tilde{K}),$$

where K is defined by (4.59), and $\tilde{K}$ is an integral operator on $L^2(\Sigma_{in} \cup \Sigma_{out}, \frac{dz}{2\pi i})$ with kernel

$$(4.70) \qquad \tilde{K}(z, w) = \begin{cases} \sqrt{\tilde{v}(z)\tilde{v}(w)} \frac{F_0(z)G_0(w) - F_0(w)G_0(z)}{w - z}, & z, w \in \Sigma_{out}, \\[2mm] \sqrt{\tilde{v}(z)\tilde{v}(w)} \frac{F_0(z)G_1(w) - F_1(w)G_0(z)}{w - z}, & z \in \Sigma_{out}, w \in \Sigma_{in}, \\[2mm] \sqrt{\tilde{v}(z)\tilde{v}(w)} \frac{F_1(z)G_0(w) - F_0(w)G_1(z)}{w - z}, & z \in \Sigma_{in}, w \in \Sigma_{out}, \\[2mm] \sqrt{\tilde{v}(z)\tilde{v}(w)} \frac{F_1(z)G_1(w) - F_1(w)G_1(z)}{w - z}, & z, w \in \Sigma_{in}. \end{cases}$$

Here

$$(4.71) \qquad \tilde{v}(z) = \frac{v(z)}{1 + v(z)} = \begin{cases} \frac{s^M}{z^M}, & z \in \Sigma_{out}, \\[2mm] \frac{z^M}{s^M}, & z \in \Sigma_{in}. \end{cases}$$

Moreover, $\tilde{K}$ is a trace class operator.

Proof. We write

$$(4.72) \qquad A(z, w) = \frac{F(z)G(w) - F(w)G(z)}{w - z},$$

and

$$(4.73) \qquad A_{ij}(z, w) = \frac{F_i(z)G_j(w) - F_j(w)G_i(z)}{w - z},$$

where $i, j \in \{0, 1\}$.

Define

$$(4.74) \qquad A_0(z, w) := \begin{cases} A_{00}(z, w), & z, w \in \Sigma_{out}, \\[2mm] A_{01}(z, w), & z \in \Sigma_{out}, w \in \Sigma_{in}, \\[2mm] A_{10}(z, w), & z \in \Sigma_{in}, w \in \Sigma_{out}, \\[2mm] A_{11}(z, w), & z, w \in \Sigma_{in}, \end{cases}$$

and $A_1(z, w) := A(z, w) - A_0(z, w)$.

The kernel A_1 has the following properties:

(A1) For any $\alpha \geq 1$,

$$(4.75) \qquad \int_{z \in \Sigma_{in} \cup \Sigma_{out}} v(z)^\alpha A_1(z, z) \frac{dz}{2\pi i} = 0.$$

(A2) For any $i = 0, 1$ and $\alpha, \beta \geq 1$,

$$(4.76) \qquad \begin{aligned} &\int_{z \in \Sigma_{in} \cup \Sigma_{out}} \int_{w \in \Sigma_{in} \cup \Sigma_{out}} v(z)^\alpha v(w)^\beta A_i(z, w) A_1(w, z) \frac{dw}{2\pi i} \frac{dz}{2\pi i} \\ &= \int_{z \in \Sigma_{in} \cup \Sigma_{out}} v(z)^{\alpha+\beta} A_i(z, z) \frac{dz}{2\pi i}. \end{aligned}$$

(A3) For any $i, j = 0, 1$ and $\alpha, \beta \geq 1$,, we have

$$(4.77) \qquad \begin{aligned} &\int_{z \in \Sigma_{in} \cup \Sigma_{out}} \int_{w \in \Sigma_{in} \cup \Sigma_{out}} v(z)^\alpha v(w)^\beta A_i(z', z) A_1(z, w) A_j(w, w') \frac{dw}{2\pi i} \frac{dz}{2\pi i} \\ &= \int_{z \in \Sigma_{in} \cup \Sigma_{out}} v(z)^{\alpha+\beta} A_i(z', z) A_j(z, w') \frac{dz}{2\pi i}. \end{aligned}$$

Proof of (A1)

It is sufficient to show

$$(4.78) \qquad \int_{\Sigma_{out}} v(z)^\alpha A_1(z, z) \frac{dz}{2\pi i} = 0, \quad \int_{\Sigma_{in}} v(z)^\alpha A_1(z, z) \frac{dz}{2\pi i} = 0.$$

By writint $A_1(z, z)$ as $F(z)G'(z) - F'(z)G(z) - F_0(z)G_0'(z) + F_0'(z)G_0(z)$, it is easy to see that $A_1(z, z)$ is bounded when $z \in \mathcal{Q}_4 \to \infty$ and that $zA_1(z, z)$ is analytic at 0. If we change the integral contour Σ_{out} to a large arc in $\mathcal{Q}_4$ with radius going to infinity, the integrand is $|v(z)^\alpha A_1(z, z)| = O(|z|^{-\alpha M}) = O(|z|^{-2})$ and then the integral can be arbitrary small. So we proved the first equation in (4.78). Similarly if we shrink Σ_{in} to 0 we immediately obtain the second equation.

Proof of (A2)

We first show

$$(4.79)$$
$$\int_{z \in \Sigma_{out}} \int_{w \in \Sigma_{out}} v(z)^\alpha v(w)^\beta A_i(z, w) A_1(w, z) \frac{dw}{2\pi i} \frac{dz}{2\pi i} = \int_{z \in \Sigma_{out}} v(x)^{\alpha+\beta} A_i(z, z) \frac{dz}{2\pi i}.$$

By using the symmetry and the analyticity of the kernel $A_i(z, w)$ ($i = 0, 1$), one can slightly deform the two contours to two nonintersecting contours $z \in \Sigma_{out,1} := \Sigma_{out} - \epsilon$ and $w \in \Sigma_{out,2} := \Sigma_{out} + \epsilon$ for some small positive number ϵ. Write $A_1(w, z) = A_{11}(w, z) + A_{1,0}(w, z) + A_{0,1}(w, z)$. The property (P2) (see (4.67)) implies

$$(4.80) \qquad \int_{w \in \Sigma_{out,2}} v(w)^\beta A_i(z, w) A_{11}(w, z) \frac{dw}{2\pi i} = 0,$$

and

$$(4.81) \qquad \int_{w \in \Sigma_{out,2}} v(w)^\beta A_i(z, w) A_{10}(w, z) \frac{dw}{2\pi i} = 0.$$

So the left hand side of (4.79) becomes

$$(4.82)
\begin{aligned}
& \int_{z \in \Sigma_{out,1}} \int_{w \in \Sigma_{out,2}} v(z)^\alpha v(w)^\beta A_i(z, w) A_{01}(w, z) \frac{dw}{2\pi i} \frac{dz}{2\pi i} \\
&= \int_{w \in \Sigma_{out,2}} \int_{z \in \Sigma_{out,1}} v(z)^\alpha v(w)^\beta A_i(z, w) A_{01}(w, z) \frac{dz}{2\pi i} \frac{dw}{2\pi i} \\
&= \int_{w \in \Sigma_{out,2}} \int_{z \in \Sigma_{out,2}+\epsilon} v(z)^\alpha v(w)^\beta A_i(z, w) A_{01}(w, z) \frac{dz}{2\pi i} \frac{dw}{2\pi i} \\
&\quad + \int_{w \in \Sigma_{out,2}} v(w)^{\alpha+\beta} A_i(w, w) \frac{dw}{2\pi i w},
\end{aligned}$$

where we used the property (P3) in the last equation. Combining with the fact

$$(4.83) \qquad \int_{z \in \Sigma_{out,2}+\epsilon} v(z)^\alpha A_i(z, w) A_{01}(w, z) \frac{dz}{2\pi i} = 0,$$

we obtain (4.79). Similarly, we have

$$(4.84)$$
$$\int_{z \in \Sigma_{in}} \int_{w \in \Sigma_{in}} v(z)^\alpha v(w)^\beta A_i(z, w) A_1(w, z) \frac{dw}{2\pi i} \frac{dz}{2\pi i} = \int_{z \in \Sigma_{in}} v(z)^{\alpha+\beta} A_i(z, z) \frac{dz}{2\pi i},$$

$$(4.85) \qquad \int_{z \in \Sigma_{in}} \int_{w \in \Sigma_{out}} v(z)^\alpha v(w)^\beta A_i(z, w) A_1(w, z) \frac{dw}{2\pi i} \frac{dz}{2\pi i} = 0,$$

and

$$(4.86) \qquad \int_{z \in \Sigma_{out}} \int_{w \in \Sigma_{in}} v(z)^\alpha v(w)^\beta A_i(z, w) A_1(w, z) \frac{dw}{2\pi i} \frac{dz}{2\pi i} = 0.$$

So we proved (A2).

Proof of (A3) It is similar to the proof of (A2).

By using the properties (A1), (A2), (A3) we can prove the following two claims.

Claim IV.18. *Both $\sqrt{v}A_0\sqrt{v}$ and $\sqrt{v}A_1\sqrt{v}$ are trace class operators. As a result, $\tilde{K}$ is also a trace class operator.*

Claim IV.19. $\det(1 + \sqrt{v}A_1\sqrt{v}) = 1$.

Together with the following Lemma and the a simple conjugation, $\det(1 + K) = \det(1 + \tilde{K})$ and Proposition IV.17 follows.

Lemma IV.20. *Let $v(z)$ be a function on $\mathfrak{C}$, a finite union of simple contours, such that*

$$(4.87) \qquad \sup_{z \in \mathfrak{C}} |v(z)| < 1.$$

Let $A = A_0 + A_1$ be the sum of two integral operators on $L^2(\mathfrak{C}, \frac{dz}{2\pi i})$ such that $\sqrt{v}A_0\sqrt{v}$ and $\sqrt{v}A_1\sqrt{v}$ are both trace class, which satisfy

$$(4.88) \qquad \operatorname{tr}[v^{n-\frac{1}{2}}A_i v^m A_1\sqrt{v}] = \operatorname{tr}[v^{n+m-\frac{1}{2}}A_i\sqrt{v}],$$

$$(4.89) \qquad A_i v^n A_1 v^m A_j = A_i v^{n+m} A_j,$$

for all $i, j \in \{0, 1\}$ and $n, m \geq 1$. Then

$$(4.90) \qquad \det(1 + \sqrt{v}A\sqrt{v}) = \det(1 + \sqrt{v}A_1\sqrt{v}) \det\left(1 + \frac{\sqrt{v}}{1+v}A_0\sqrt{v}\right).$$

The rest of the proof is to show Lemma IV.20, Claim IV.18 and Claim IV.19.

Proof of Lemma IV.20 We write the right hand side of (4.90) as $\det(1 + X + Y)$ where $X = \sqrt{v}A_1\sqrt{v} + \frac{\sqrt{v}}{1+v}A_0\sqrt{v}$, $Y = \sqrt{v}A_1\frac{v}{1+v}A_0\sqrt{v}$. Denote $\tilde{Y} = \frac{v\sqrt{v}}{1+v}A_0\sqrt{v}$. We can check that $X, Y, \tilde{Y}$ have the following properties:

(a) $\text{tr}[Y^n] = \text{tr}[\tilde{Y}^n]$, for all $n \geq 1$,

(b) $\text{tr}[XY^n] = \text{tr}[X\tilde{Y}^n]$, for all $n \geq 1$,

(c) $XY^nX = X\tilde{Y}^nX$, for all $n \geq 1$.

By applying these properties, it is easy to check $\text{tr}(X + Y)^n = \text{tr}(X + \tilde{Y})^n$ for all $n \geq 1$. Therefore $\det(1 + X + Y) = \det(1 + X + \tilde{Y})$. Combining with the fact that $1 + X + \tilde{Y} = 1 + \sqrt{v}A\sqrt{v}$, (4.90) follows immediately.

Now we come back to check (a), (b) and (c). Notice that (4.88) and (4.90) imply

$$(4.91) \qquad \text{tr}[f(v)v^{-\frac{1}{2}}A_i g(v)A_1\sqrt{v}] = \text{tr}[f(v)g(v)v^{-\frac{1}{2}}A_i\sqrt{v}],$$

$$(4.92) \qquad A_i f(v)A_1 g(v)A_j = A_i f(v)g(v)A_j,$$

for all $i, j \in \{0, 1\}$, where $f(z), g(z)$ are two arbitrary functions which have convergent power series expansion within the unit disk and satisfy $f(0) = g(0) = 0$. Especially we apply (4.91) by taking $f(v) = \frac{v}{1+v}$, $g(v) = v$ and $i = 0$, and obtain $\text{tr}(Y) = \text{tr}(\tilde{Y})$. For $n \geq 2$, we apply (4.92) and have

$$(4.93) \qquad \begin{aligned} Y^n &= \sqrt{v}A_1\frac{v}{1+v}A_0 vA_1\frac{v}{1+v}A_0 \cdots vA_1\frac{v}{1+v}A_0\sqrt{v} \\ &= \sqrt{v}A_1\frac{v}{1+v}A_0\frac{v^2}{1+v}A_0 \cdots \frac{v^2}{1+v}A_0\sqrt{v}. \end{aligned}$$

Then we use the identity $\text{tr}(AB) = \text{tr}(BA)$ and have

$$(4.94) \qquad \begin{aligned} \text{tr}(Y^n) &= \text{tr}\left(\frac{v\sqrt{v}}{1+v}A_0 vA_1\frac{v}{1+v}A_0\frac{v^2}{1+v}A_0 \cdots \frac{v^2}{1+v}A_0\sqrt{v}\right) \\ &= \text{tr}\left(\frac{v\sqrt{v}}{1+v}A_0\frac{v^2}{1+v}A_0 \cdots \frac{v^2}{1+v}A_0\frac{v^2}{1+v}A_0\sqrt{v}\right), \end{aligned}$$

where we used (4.92) again in the second equation. Therefore we proved (a) for all n. Similarly we can prove (b) and (c).

Proof of Claim IV.18   The strategy is to write $\sqrt{v}A_0\sqrt{v}$ as the product of two operators which are both Hilbert-Schmidt. We first define two new contours Σ'_{in}

and Σ'_{out} as follows. They satisfy the conditions (C1), (C2), and (C3), where the parameter λ for the condition (C2) is replaced by λ' for some $\lambda' \in (\lambda, 2)$. Moreover, we take Σ'_{in} to be between Σ_r and Σ_{in}, and Σ'_{out} to be between Σ_r and Σ_{out}.

Note that it is possible to take such Σ'_{out} since $\lambda < \lambda'$. This condition, together with (C2), also implies that

$$\text{(4.95)} \qquad \text{dist}(w, \Sigma'_{out}) = O(\log |w|),$$

as $w \to \infty$ satisfying $w \in \Sigma_{out}$.

Define the operator L_1 mapping $L^2(\Sigma'_{in} \cup \Sigma'_{out}, \frac{dz}{2\pi i})$ to $L^2(\Sigma_{in} \cup \Sigma_{out}, \frac{dz}{2\pi i})$ by the kernel

$$\text{(4.96)} \qquad L_1(z, w) = \begin{cases} \sqrt{v(z)}A_{11}(z, w), & z \in \Sigma_{in}, w \in \Sigma'_{in}, \\[2mm] \sqrt{v(z)}A_{10}(z, w)\sqrt{v(w)}w, & z \in \Sigma_{in}, w \in \Sigma'_{out}, \\[2mm] \sqrt{v(z)}A_{01}(z, w), & z \in \Sigma_{out}, w \in \Sigma'_{in}, \\[2mm] \sqrt{v(z)}A_{00}(z, w)\sqrt{v(w)}w, & z \in \Sigma_{out}, w \in \Sigma'_{out}. \end{cases}$$

Similarly, define the operator L_2 mapping $L^2(\Sigma_{in} \cup \Sigma_{out}, \frac{dz}{2\pi i})$ to $L^2(\Sigma'_{in} \cup \Sigma'_{out}, \frac{dz}{2\pi i})$ by the kernel

$$\text{(4.97)} \qquad L_2(z, w) = \begin{cases} A_{01}(z, w)\sqrt{v(w)}, & z \in \Sigma'_{in}, w \in \Sigma_{in}, \\[2mm] A_{00}(z, w)\sqrt{v(w)}, & z \in \Sigma'_{in}, w \in \Sigma_{out}, \\[2mm] v(z)^{-1/2}z^{-1}A_{11}(z, w)\sqrt{v(w)}, & z \in \Sigma'_{out}, w \in \Sigma_{in}, \\[2mm] v(z)^{-1/2}z^{-1}A_{10}(z, w)\sqrt{v(w)}, & z \in \Sigma'_{out}, w \in \Sigma_{out}. \end{cases}$$

Then

$$\text{(4.98)} \qquad \sqrt{v(z)}A_0(z, w)\sqrt{v(w)} = \int_{\Sigma'_{in} \cup \Sigma'_{out}} L_1(z, u)L_2(u, w)\frac{du}{2\pi i},$$

for all $z, w \in \Sigma_{in} \cup \Sigma_{out}$. This can be shown as follows when $z, w \in \Sigma_{out}$. In this case, the right hand side of (4.98) equals $\sqrt{v(z)v(w)}$ times

$$(4.99) \qquad \int_{\Sigma'_{in}} A_{01}(z, u) A_{00}(u, w) \frac{du}{2\pi i} + \int_{\Sigma'_{out}} A_{00}(z, u) A_{10}(u, w) \frac{du}{2\pi i}.$$

Now, $A_{01}(z, u) A_{00}(u, w)$ is analytic for u inside Σ'_{in} since $z, w \in \Sigma_{out}$, hence the first integral is 0. On the other hand, for $z, w \in \Sigma_{out}$, $A_{00}(z, u) A_{10}(u, w)$ is analytic for u on the right of the contour Σ'_{out}, except for the simple pole at $u = w$. By using (4.68) one can compute the residue at $u = w$ and rewrite (4.99) as

$$(4.100) \qquad 0 + A_{00}(z, w) + \int_{C_R} A_{00}(z, u) A_{10}(u, w) \frac{du}{2\pi i}$$

where R is some positive number larger than $|w|$, and C_R is some contour outside of $\{z; |z| = R\}$ which satisfies (4.18). Finally, taking $R \to \infty$ and noting that $|A_{00}(z, u) A_{10}(u, w)| = O(|u|^{-2})$, we find that (4.98) holds when $z, w \in \Sigma_{out}$. For other cases of z and w the proof is similar and we skip the details.

Now we show that

$$(4.101) \qquad \int_{\Sigma'_{in} \cup \Sigma'_{out}} \int_{\Sigma_{in} \cup \Sigma_{out}} |L_1(z, w)|^2 |dz||dw| < \infty,$$

and

$$(4.102) \qquad \int_{\Sigma_{in} \cup \Sigma_{out}} \int_{\Sigma'_{in} \cup \Sigma'_{out}} |L_2(z, w)|^2 |dz||dw| < \infty.$$

If $z \in \Sigma_{in}, w \in \Sigma'_{out}$, we have $|L_1(z, w)| = O(|w|^{-\frac{\lambda'}{2}})$ as w becomes large by condition (C2) and the fact that $n\phi(w) = \frac{T}{2}w - \frac{a-n}{2} \log w + O(1)$ as $w \to \infty$. Since $\lambda' > \lambda > 1$, $L_1(z, w)$ is square integrable on $\Sigma_{in} \times \Sigma'_{out}$. Similarly if $z \in \Sigma_{out}, w \in \Sigma'_{in}$, we have $|L_1(z, w)| = O(|z|^{-1-\frac{\lambda}{2}})$. Hence (4.101) follows if we show that

$$(4.103) \qquad \int_{\Sigma'_{out}} \int_{\Sigma_{out}} |L_1(z, w)|^2 |dz||dw| < \infty.$$

For this purpose, we observe that

$$(4.104) \qquad \frac{e^{-n(\phi(z)+\phi(w))}(F_0(z)G_0(w) - F_0(w)G_0(z))}{w - z} = O(|zw|^{-1})$$

when $z, w \in \mathcal{Q}_4$ tend to infinity. This follows from (4.62), (4.63), the properties $R(z) = I + O(|z|^{-1})$ and $\beta(z) = 1 + O(|z|^{-1})$. As a result $|L_1(z, w)| = O(|z|^{-1-\frac{\lambda}{2}}|w|^{-\frac{\lambda'}{2}})$ for $z \in \Sigma_{out}, w \in \Sigma'_{out}$. This implies (4.103) and hence (4.101) is proved.

We now prove (4.102). If $z \in \Sigma'_{out}, w \in \Sigma_{in}$, a calculation as before implies that

$$(4.105) \qquad |L_2(z, w)| = O(|z|^{\frac{\lambda'-3}{2}}).$$

Since $\lambda' < 2$, $L_2(z, w)$ is square integrable on $\Sigma'_{out} \times \Sigma_{in}$. Similarly if $z \in \Sigma'_{in}, w \in \Sigma_{out}$, we have $L_2(z, w) = O(|w|^{-1})$. Finally, consider the case when $z \in \Sigma'_{out}, w \in \Sigma_{out}$. The estimates on this case is more delicate. We have $|L_2(z, w)| \leq C|z|^{\frac{\lambda'}{2}-1}|w|^{-\frac{\lambda}{2}}|z - w|^{-1} \leq C|w|^{-\frac{\lambda}{2}}|z - w|^{-1}$ for some constant C. Notice that if $w \in \Sigma_{out}$ large enough, we have

$$(4.106) \qquad \begin{aligned} &\int_{\Sigma'_{out} \cap \{z; |\Im(z)-\Im(w)| \leq \log|w|\}} |z - w|^{-2}|dz| \\ &\leq \int_{\Sigma'_{out} \cap \{z; |\Im(z)-\Im(w)| \leq \log|w|\}} \frac{C_1}{(\log|w|)^2}|dz| \\ &\leq \frac{C'_1}{\log|w|}, \end{aligned}$$

where we used the facts that $\text{dist}(w, \Sigma'_{out}) = O(\log|w|)$ and $\Im(z) \sim z + O(\log|z|)$ by the condition (C2) of Σ'_{out}. We also have

$$(4.107) \qquad \begin{aligned} &\int_{\Sigma'_{out} \cap \{z; |\Im(z)-\Im(w)| > \log|w|\}} |z - w|^{-2}|dz| \\ &\leq \int_{\Sigma'_{out} \cap \{z; |\Im(z)-\Im(w)| > \log|w|\}} |\Im(z) - \Im(w)|^{-2}|dz| \\ &\leq C_2 \int_{\Sigma'_{out} \cap \{z; |\Im(z)-\Im(w)| > \log|w|\}} |\Im(z) - \Im(w)|^{-2}d\Im(z) \\ &= \frac{C'_2}{\log|w|}. \end{aligned}$$

Therefore, by combing (4.106) and (4.107), we have

$$(4.108) \qquad \int_{\Sigma'_{out}} |z - w|^{-2}|dz| \leq \frac{C_3}{\log |w|}$$

for $w \in \Sigma_{out}$ large enough, where C_3 is some positive constant. Hence

$$(4.109) \qquad \int_{\Sigma_{out}} \int_{\Sigma'_{out}} |L_2(z, w)|^2 |dz||dw| < \infty$$

by the square integrability of $|w|^{-\lambda/2}(\log |w|)^{-1/2}$ on Σ_{out}. (4.102) is proved by the discussions above for the three cases.

Now $\sqrt{v}A_0\sqrt{v}$ is a composition of two Hilbert-Schmidt operators, hence it is a trace class operator. Notice that K itself is a trace class operator (see Remark IV.7), $\sqrt{v}A_1\sqrt{v} = K - \sqrt{v}A_0\sqrt{v}$ is also a trace class operator. $\tilde{K}$ is the multiplication of some bounded operators and a trace class operator $\sqrt{v}A_0\sqrt{v}$, so it is trace class.

Proof of Claim IV.19 By expanding the Fredholm determinant $\det(1 + \sqrt{v}A_1\sqrt{v})$ and using the properties (A1),(A2),(A3) of the kernel A_1, one obtain Claim IV.19 immediately. $\square$

4.2.2 Steepest Descent Analysis

In this section, we will use Proposition IV.17 to prove Theorem IV.3. We suppose $T = O(n)$ (see the Remark IV.2) and $a = (1 + \gamma)n$ for some positive constant γ. Set $M = 2\sqrt{an} + x\frac{(a-n)^{2/3}}{2^{2/3}(an)^{1/6}}$ where $x \in \mathbb{R}$ is fixed. We will consider the asymptotics of $\det(1 + \tilde{K})$ when $n \to \infty$.

First we notice that we can remove the condition that Σ_{in}, Σ_{out} are inside, outside respectively, of the circle Σ_r. In fact, $\tilde{K}$ has no poles on Σ_r. $\text{tr}(\tilde{K}^k)$ $(k = 0, 1, \cdots)$, is independent of whether $\Sigma_{in} \cup \Sigma_{out}$ intersects Σ_r or not. Moreover, if we change the contours in some bounded region, $\tilde{K}$ is still a trace class operator as long as the two contours does not intersect in that region. This follows from a similar argument with the claim that $\sqrt{v}A_0\sqrt{v}$ is a trace class, where we just need to replace v by $\tilde{v}$.

In order to find the steepest descent contour of the kernel, we need to consider the leading terms of $R(z)$, $\phi(z)$ and $\tilde{v}(z)$.

For $R(z)$, we come back to the Riemann-Hilbert problem IV.16. Notice that the jump matrix is exponentially small except at a small neighborhood of ξ and $\bar{\xi}$. Moreover, the difference of the jump matrix v_R and I decays fast enough as it behaves like $z^{-M/2}$. By using the standard Riemann-Hilbert analysis([32]), we know

$$(4.110) \qquad R(z) = I + O(n^{-1})$$

uniformly for z with $O(1)$ distance to the jump contours of $R(z)$, i.e., uniformly for z satisfying $\text{dist}(z, \Gamma' \cup \Gamma_{in} \cup \Gamma_{out,1} \cup \Gamma_{out,2}) \geq C$ where C is some positive constant.

We define

$$(4.111) \qquad \psi(z) := \frac{M}{2n} \left(\log(z) - \log(s) \right).$$

Therefore

$$(4.112) \qquad \tilde{v}(z) = \begin{cases} e^{2n\psi(z)}, & z \in \Sigma_{in}, \\ \\ e^{-2n\psi(z)}, & z \in \Sigma_{out}. \end{cases}$$

Denote by $\psi_c(z) = \sqrt{a/n}(\log(z) - \log(s))$ the leading term of $\psi(z)$. Then $z_c := \Re(\xi) = \frac{a-n}{T}$ is the double critical point of the function $\phi(z) + \psi_c(z)$. In fact,

$$
\begin{aligned}
\frac{d}{dz} \left(\phi(z) + \psi_c(z) \right) &= \frac{T}{2n} \frac{\sqrt{(z - \xi)(z - \bar{\xi})}}{z} + \frac{2\sqrt{an}}{2nz} \\
&= \frac{T}{2nz} \left(\sqrt{(z - z_c)^2 + \Im(\xi)^2} + \Im(\xi) \right) \\
&= \frac{T(z - z_c)^2}{2nz \left(\sqrt{(z - z_c)^2 + \Im(\xi)^2} - \Im(\xi) \right)} \\
&= -\frac{T^3}{8a^{1/2}n^{3/2}(a - n)} (z - z_c)^2 + O(|z - z_c|^3)
\end{aligned}
$$

(4.113)

as $z \to z_c$. Here the notation $O(|z - z_c|^3)$ means the term is bounded by $C|z - z_c|^3$ where C is independent of both z_c and n.

Now we pick

(4.114)
$$r = |s| = z_c e^{\frac{2n}{M}\Re(\phi(z_c))}.$$

Then

(4.115)
$$\Re\left(\phi(z_c) + \psi(z_c)\right) = 0.$$

By using (4.113) we have the following expansion in a small (but still $O(1)$ with respect to n) neighborhood of z_c

(4.116)
$$\phi(z) + \psi(z) = ic_1 + \frac{xT}{2^{5/3}a^{1/6}n^{7/6}(a-n)^{1/3}}(z - z_c) - \frac{T^3}{24a^{1/2}n^{3/2}(a-n)}(z - z_c)^3$$
$$+ O(n^{-2/3}|z - z_c|^2) + O(|z - z_c|^4),$$

where c_1 is some real constant.

Step 1: Selection of the Contours

Throughout this step, we use the notation c to represent a positive constant which is independent of N and z. This constant can have different values in different places.

Denote by L the open line segment from $\bar{\xi}$ to ξ.

Recall that the set $\{z | \Re(\phi(z)) = 0\}$ consists of four segments. The segment which intersects the interval $(0, +\infty)$ and the segment which intersects $(-\infty, 0)$ are denoted by Γ and Γ_1 respectively. The union of the other two segments is denoted by Γ_2.

We now show that Γ does not intersect L, and, moreover, Γ lies on the right of L. Suppose that Γ intersects L. Let $z_c + i\lambda$ be the intersection with the largest imaginary value. Then $\Re(\phi(z_0 + i\lambda)) = 0$ since $z_0 + i\lambda \in \Gamma$. On the other hand,

(4.117)
$$\Re(\phi(z_c + i\lambda)) = \int_{\Im(\xi)}^{\lambda} \frac{d}{dt}\Re(\phi(z_c + it))dt$$
$$= \frac{T}{2n}\int_{\Im(\xi)}^{\lambda} \frac{\sqrt{(z_c + it - \xi)(z_c + it - \bar{\xi})}}{t}dt.$$

Notice that $\sqrt{(z_c + it - \xi)(z_c + it - \bar{\xi})} = \sqrt{\Im(\xi)^2 - t^2}$ is real for $t \in [-\Im(\xi), \Im(\xi)]$. This is positive if $\tilde{\Gamma}$, the part of Γ from ξ to $z_c + i\lambda$, is to the left of L, and is negative if $\tilde{\Gamma}$ is to the right of L. In any case, we see that $\Re(\phi(z_0 + i\lambda)) \neq 0$. This gives a contradiction. Hence Γ does not intersect L. Now $\Re(\phi(z_c)) > 0$ since z_c lies to the right of Γ_1 and $z_c \in L$. So $\sqrt{(z_c + it - \xi)(z_c + it - \bar{\xi})} < 0$ for all $t \in (0, \Im(\xi))$, which implies Γ is on the right of L. Thus, the intersection of Γ and $\mathbb{R}$ is on $(z_c, +\infty)$. See Figure 4.5.

Denote by $\tilde{\phi}(z)$ the new function obtained by changing the branch cut of $\phi(z)$ to Γ_2, i.e., $\tilde{\phi}(z) := -\phi(z)$ if z in the region which contains $+\infty$ and is bounded by Γ and Γ_2, and $\tilde{\phi}(z) = \phi(z)$ for all other z. See Figure 4.3. Recall that Γ_2 does not intersect the region $\mathcal{Q}_3 \cup \mathcal{Q}_4$. This fact makes $\tilde{\phi}(z)$ more convenience than $\phi(z)$ since our contours Σ_{in} and Σ_{out} will be chosen in $\mathcal{Q}_3 \cup \mathcal{Q}_4$.
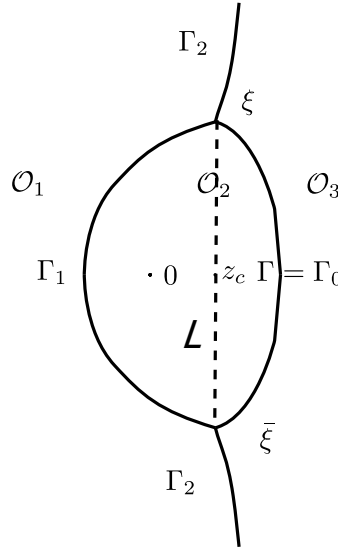

Figure 4.5: The graph of $\Gamma \cup \Gamma_1 \cup \Gamma_2 = \{z | \Re(\phi(z)) = 0\}$ and L

.

We first discuss the global picture of the set $\{z | \Re(\tilde{\phi}(z) + \psi(z)) = 0\}$ and its relative location to the set $\{z | \Re(\phi(z)) = 0\}$.

Note that

$$(4.118) \quad \frac{d(\Re(\tilde{\phi}(z) + \psi(z)))}{dz} = \frac{T\sqrt[2]{(z - \xi)(z - \bar{\xi})} + M}{2nz}$$

$$= \frac{T}{2n} \frac{\sqrt[2]{(z - \xi)(z - \bar{\xi})} + \Im(\xi) + O(n^{-2/3})}{z},$$

where the $\sqrt[2]{}$ denotes the square root function which is negative on $\mathbb{R}$ and is branch cut is Γ_2. It is easy to see that $\Re(\tilde{\phi} + \psi)(\pm\infty) = -\infty$, and $\Re(\tilde{\phi} + \psi)(0) = +\infty$. Moreover, $\Re(\tilde{\phi} + \psi)$ is monotonic in $(-\infty, 0)$ and in $(0, +\infty)$.

For $z \in L$,

$$\phi(z) + \psi(z) - \phi(z_c) - \psi(z_c)$$

$$(4.119) \quad = \int_{z_c}^{z} \frac{T\sqrt{(w - \xi)(w - \bar{\xi})} + M}{2nw} dw$$

$$= \int_0^{\Im(z)/\Im(\xi)} \frac{-T\Im(\xi)\sqrt{1 - t^2} + M}{2n(z_c + t\Im(\xi)i)} i\Im(\xi) dt,$$

Since $\Re(\phi(z_c) + \psi(z_c)) = 0$, $M = T\Im(\xi) + O(n^{1/3})$, and $T = O(n)$, we find that

$$(4.120) \quad \Re(\phi(z) + \psi(z)) = \int_0^{\Im(z)/\Im(\xi)} \frac{-T\Im(\xi)\sqrt{1 - t^2} + M}{2n(z_c^2 + t^2\Im(\xi)^2)} t\Im(\xi)^2 dt$$

$$= \frac{T\Im(\xi)^3}{2n} \int_0^{|\Im(z)|/\Im(\xi)} \frac{-\sqrt{1 - t^2} + 1 + O(nN^{-2/3})}{z_c^2 + t^2\Im(\xi)^2} |t| dt.$$

Now it is easy to check that for any $\epsilon > 0$, there exists $n_0 > 0$ such that

$$(4.121) \quad \int_0^{\epsilon} \frac{-\sqrt{1 - t^2} + 1 + O(n^{-2/3})}{z_c^2 + t^2\Im(\xi)^2} |t| dt > 0$$

for all $n \geq n_0$. Therefore we find that $\forall \epsilon > 0$, there exists $n_0 > 0$ such that

$$(4.122) \quad \Re(\tilde{\phi}(z) + \psi(z)) = \Re(\phi(z) + \psi(z)) > 0,$$

for all $z \in L$ such that $|z - z_c| \geq \epsilon$ and for all $n \geq n_0$.

Now we claim that there are at most four points in the intersection of $\Gamma \cup \Gamma_1 \cup \Gamma_2 = \{z | \Re(\phi(z)) = 0\}$ and $\{z | \Re(\tilde{\phi}(z) + \psi(z)) = 0\}$. See Figure 4.7. Note that the intersection is the same as that of $\Re(\phi(z)) = 0$ and $\Re(\psi(z)) = 0$. Note that

$\{z|\Re(\psi(z)) = 0\}$ is the circle centered at the origin with radius r where r is defined in (4.114). We have $r > z_c$ by (4.115) and $\Re(\phi(z_c)) > 0$, and also $r < |\xi|$ by (4.122) and $\Re(\phi(\xi)) = 0$. Consider the function $H(\theta) := \Re(\phi(re^{i\theta}))$. This function is continuous at all θ but the derivative is discontinuous at θ if $re^{i\theta} \in \Gamma$. It is easy to check that

$$(4.123) \qquad \frac{dH(\theta)}{d\theta} = -\Im\left(\sqrt{(re^{i\theta} - \xi)(re^{i\theta} - \bar{\xi})}\right).$$

Notice that $(re^{i\theta} - \xi)(re^{i\theta} - \bar{\xi})$ is real only if $\theta = 0, \pi, \pm\arccos(\Re(\xi)/r)$, which means (4.123) has at most four zeros. These zeros divide the circle into at most four arcs. Since $H(\theta) = 0$ at the non-differential points, one can see that each closed arc contains at most one zero of $H(\theta)$. Hence, there are at most four points on the intersection of $\{z|\Re(\phi(z)) = 0\}$ and $\{z|\Re(\tilde{\phi}(z) + \psi(z)) = 0\}$.

Now we are ready to sketch a graph of the set $\{z|\Re(\tilde{\phi}(z) + \psi(z)) = 0\}$. This set can be viewed as the integral curve of the vector field $\Re((\tilde{\phi}'(z) + \psi'(z))dz) = 0$. Note that the vector field has a pole at 0, a critical point at z_c, and it is discontinuous along Γ_2. By the discussion after (4.118), $\{z|\Re(\tilde{\phi}(z) + \psi(z)) = 0\}$ intersects $\mathbb{R}$ at exact two poins which are z_c and a point on the left of the origin. Moreover, the vector field is circular near 0. When $z \to z_c$, by using (4.116), there are six rays of $\Re(\tilde{\phi} + \psi) = 0$ going out z_c with angles $\pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$. Each pair of neighboring rays form a region, (if they intersect, we choose the first bounded region), we label these regions from left to right the region $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_4$. See Figure 4.6 for example. Here by symmetry the region $\mathcal{R}_2$ and $\mathcal{R}_3$ both represent he union of two regions formed by the neighboring rays. Note that it might be possible that two regions turn out to be the same one. For example, if the two rays with angles $\frac{\pi}{2}$ and $\frac{\pi}{6}$ meet, the regions $\mathcal{R}_2$ and $\mathcal{R}_4$ are the same. The cubic behavior of $\tilde{\phi} + \psi$ near z_c implies that $\Re(\tilde{\phi} + \psi) > 0$ in the region $\mathcal{R}_1$ and $\mathcal{R}_3$, and that $\Re(\tilde{\phi} + \psi) < 0$ in the region $\mathcal{R}_2$ and

$\mathcal{R}_4$.

By symmetry and a standard topological argument, the region I is bounded and contains the origin.

By (4.122), L is contained the region $\mathcal{R}_1$ or $\mathcal{R}_3$. However, L is outside of the region $\mathcal{R}_1$ near z_c. It implies that L is contained in the region $\mathcal{R}_3$. Since the boundary rays of the region $\mathcal{R}_3$ cannot intersect $\mathbb{R}$ except z_c, both of Γ and Γ_1 intersect the boundary of the region $\mathcal{R}_3$. Hence we obtained at least four intersection points of $\{z|\Re(\phi(z)) = 0\}$ and $\{z|\tilde{\phi}(z) + \psi(z) = 0\}$. By the discussion about the maximal number of intersection points in the paragraph before (4.123), there are exact four intersection points. Two points are on Γ and the other two are on Γ_1. Therefore the trajectories of $\{z|\Re(\tilde{\phi}(z) + \psi(z)) = 0\}$ do not intersect Γ_2. We can complete the six rays going out from z_c and obtain the Figure 4.6. These six rays actually are all the trajectories of $\{z|\Re(\tilde{\phi}(z) + \psi(z)) = 0\}$. In fact, by a similar argument to our previous discussion one can show there are at most four intersection points of $\{z|\Re(\tilde{\phi}(z) + \psi(z)) = 0\}$ and $\{z||z| = R\}$ for any $R \geq |\xi|$. On the other hand, the four unbounded rays in Figure 4.6 intersect with $\{z||z| = R\}$ at exact four points. For $R \in (r, |\xi|)$, we similarly have the formula (4.123) with r replaced by R. Therefore for each arc divided by the critical points, Γ_2 cannot intersect twice since the function is monotonic between two nearest intersecting points. We also notice that Γ_2 is bounded by $\Gamma_1 \cup \Gamma$ and the rays going out z_c with angles $\pm\frac{\pi}{2}, \pm\frac{\pi}{6}$, therefore Γ_2 cannot intersect the arc (divided by the critical points) at only one point. Hence Γ_2 does not intersect $\{z||z| = R\}$ when $R \in (r, |\xi|)$, and there are exact four intersection points of $\{z|\Re(\tilde{\phi}(z) + \psi(z)) = 0\}$ and $\{z||z| = R\}$, which are on the rays in Figure 4.6.

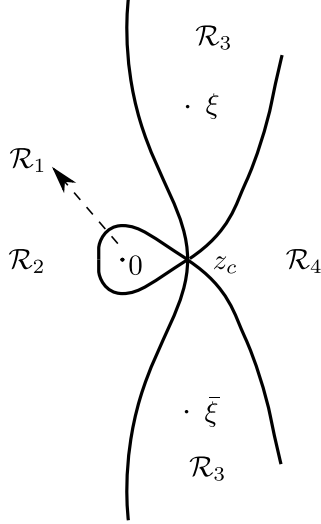Now we are ready to choose the contours Σ_{in} and Σ_{out}. We pick the contour Σ_{in}

Figure 4.6: The graph of $\Re(\tilde{\phi}(z) + \psi(z)) = 0$

Figure 4.7: New choice of Σ_{in} and Σ_{out}

in the intersection of $\mathcal{Q}_3$ and the region $\mathcal{R}_2$, and the contour Σ_{out} in the intersection of $\mathcal{Q} = \mathcal{Q}_3 \cup \mathcal{Q}_4$ and the region $\mathcal{R}_3$. See Figure 4.7. Then the kernel $\tilde{K}$ decays exponentially except near the double critical point z_c. More precisely, we construct the contours $\Sigma_{out} = \Sigma_{out,0} \cup \Sigma_{out,1}$ and $\Sigma_{in} = \Sigma_{in,0} \cup \Sigma_{in,1}$ satisfying the following conditions.

(1) $\Sigma_{in} \subset \mathcal{Q}_3$, $\Sigma_{out} \subset \mathcal{Q}_3 \cup \mathcal{Q}_4$, and $\text{dist}(\Sigma_{in} \cup \Sigma_{out}, \Gamma_{in} \cup \Gamma_{out,1} \cup \Gamma_{out,2}) > c$,

(2) $\Sigma_{in,1}$ and $\Sigma_{out,1}$ are both in a disc centered at z_c of radius $cn^{-1/4}$.

(3) $\Re(-\phi(z) + \psi(z)) > c$, for all $z \in \Sigma_{out,2} \cap \mathcal{Q}_4$. Moreover, $\Re(-\phi(z) + \psi(z) - \frac{\lambda}{2n} \log z) = c$ when $z \in \Sigma_{out,2}$ goes to infinity.

(4) $\Re(\phi(z) + \psi(z)) > cn^{-3/4}$ for all $z \in \Sigma_{out,2} \cap \mathcal{Q}_3$.

(5) $\Re(\phi(z) + \psi(z)) < -cn^{-3/4}$ for all $z \in \Sigma_{in,2}$.

The condition (1) and the second part of (3) are basically (C3) and (C2), which are necessary to apply Proposition IV.17.

Now we pick $\Sigma_{in,1}$ to be a smooth curve in $\{z; |z - z_c| \leq cn^{-1/4}\}$ which behaves like $\arg(z - z_c) = \pm\frac{2\pi}{3}$ as $|z - z_c| \geq O(n^{-1/3})$. Notice that $\phi(z) + \psi(z) \sim ic_1 - c(z - z_c)^3$

as $n^{-1/3} \leq |z - z_c| \leq n^{-1/4}$ by (4.116), therefore the endpoints of $\Sigma_{in,1}$ is still in the region $\mathcal{R}_1$. We can construct $\Sigma_{in} = \Sigma_{in,2} \cup \Sigma_{in,1}$ by completing $\Sigma_{in,1}$ smoothly in the region $\mathcal{R}_1$ such that the condition (5) holds.

Similarly we can pick $\Sigma_{out,1}$ and $\Sigma_{out,2}$ in the region $\mathcal{R}_2$ satisfying the conditions (1)-(4).

Step 2: Asymptotic Analysis

Now we are going to discuss the asymptotics of $\check{K}(z, w)$ on $\Sigma_{in} \cup \Sigma_{out}$. Note the following:

(1) For all $z \in \Sigma_{out,2} \cap \mathcal{Q}_3$,

$$(4.124) \qquad e^{-n\psi(z)} F_0(z) = O(e^{-cn^{1/4}}).$$

For all $z \in \Sigma_{out,2} \cap \mathcal{Q}_4$,

$$(4.125) \qquad e^{-n\psi(z)} F_0(z) = O(e^{-cn}).$$

Moreover, for z large enough on $\Sigma_{out,2} \cap \mathcal{Q}_4$,

$$(4.126) \qquad e^{-n\psi(z)} F_0(z) = O(e^{-cn} z^{-\frac{\lambda}{2}}).$$

(2) For all $z \in \Sigma_{in,2}$,

$$(4.127) \qquad e^{n\psi(z)} F_1(z) = O(e^{-cn^{1/4}}).$$

The above estimate is also valid if we replace F_1 by G_1.

(3) For all $z \in \Sigma_{out,1}$,

$$(4.128) \qquad e^{-n\psi(z)} F_0(z) = O(1).$$

(4) For all $z \in \Sigma_{in,1}$,

$$(4.129) \qquad e^{n\psi(z)} F_1(z) = O(1).$$

(5) The above estimations are also valid if we replace F_0 by G_0, or replace F_1 by G_1.

Therefore it is easy to check if any of z, w is in $\Sigma_{in,2} \cup \Sigma_{out,2}$, we have

$$(4.130) \qquad \check{K}(z, w) = O(e^{-cn^{1/4}}).$$

By a standard argument of Fredholm analysis, we have

$$(4.131) \qquad \det(1 + \check{K})|_{L^2(\Sigma_{in} \cup \Sigma_{out})} = \det(1 + \check{K})|_{L^2(\Sigma_{in,1} \cup \Sigma_{out,1})} + O(e^{-cn^{1/4}}).$$

Now we want to focus on $\check{K}(z, w)$ when z, w are both in $\Sigma_{in,1} \cup \Sigma_{out,1}$. We do the following rescaling

$$(4.132) \qquad \begin{aligned} z &= z_c + 2^{2/3} a^{1/6} n^{1/6} (a - n)^{1/3} T^{-1} \xi, \\ w &= z_c + 2^{2/3} a^{1/6} n^{1/6} (a - n)^{1/3} T^{-1} \eta, \end{aligned}$$

where ξ, η are both in $\tilde{\Sigma}_{in}^{(n)} := 2^{-2/3} a^{-1/6} n^{-1/6} (a - n)^{-1/3} T(\Sigma_{in,1} - z_c)$ and $\tilde{\Sigma}_{out}^{(n)} := 2^{-2/3} a^{-1/6} n^{-1/6} (a - n)^{-1/3} T(\Sigma_{out,1} - z_c)$. Then

$$(4.133) \qquad \det(1 + \check{K}(z, w)) = \det(1 + \hat{K}(\xi, \eta)),$$

where

$$(4.134) \qquad \hat{K}(\xi, \eta) = 2^{2/3} a^{1/6} n^{1/6} (a - n)^{1/3} T^{-1} \check{K}(z, w).$$

We also notice that

$$(4.135) \qquad \beta(z_c + 2^{2/3} a^{1/6} n^{1/6} (a - n)^{1/3} T^{-1} \xi) = e^{i\frac{\pi}{4}} + O(n^{-1/12}),$$

for all $\xi \in \tilde{\Sigma}_{in}^{(n)} \cup \tilde{\Sigma}_{out}^{(n)}$.

Then it is easy to check if $\xi, \eta \in \tilde{\Sigma}_{in}^{(n)}$, or $\xi, \eta \in \tilde{\Sigma}_{out}^{(n)}$ we have

$$(4.136) \qquad \hat{K}(\xi, \eta) = O(n^{-1/12}).$$

If $z \in \tilde{\Sigma}_{in}^{(n)}$ but $w \in \tilde{\Sigma}_{out}^{(n)}$, we have

$$(4.137) \qquad \hat{K}(\xi, \eta) = \frac{e^{-m_x(\xi)+m_x(\eta)}}{\eta - \xi} + O(n^{-1/12}),$$

where

$$(4.138) \qquad m_x(\xi) := -\frac{1}{2}x\xi + \frac{\xi^3}{6}.$$

Similarly, if $z \in \tilde{\Sigma}_{out}^{(n)}$ but $w \in \tilde{\Sigma}_{in}^{(n)}$, we have

$$(4.139) \qquad \hat{K}(\xi, \eta) = \frac{-e^{m_x(\xi)-m_x(\eta)}}{\eta - \xi} + O(n^{-1/12}).$$

Therefore $\det(1 + \hat{K})$ converges to $F_{GUE}(x)$ and Theorem IV.3 follows, as what we discussed at the end of subsection 3.3.

CHAPTER V

Identities on the Airy Process and Tracy-Widom Distributions

5.1 Introduction and Results

It is believed that the Airy process $\mathcal{A}(t)$ is an universal limit of the spatial fluctuations for models in the KPZ class. The limit theorem to the Airy process is established for several special cases in 2-dimensional directed last passage percolation (DLPP), 1+1 dimensional random growth, nonintersecting processes, and random matrices. See, for example, [27] and the references therein.

The basic connection between the Airy process and the Tracy-Widom distribution is that the marginal distribution of $\mathcal{A}(t)$ at a fixed t is $F_{GUE}(x)$. The joint distribution at finitely many times is also explicit and is given by a determinantal formula involving the Airy function. The Airy process is stationary but is not Markovian.

In addition to the above basic connection, there are interesting identities between the supremum of a function of the Airy process and the Tracy-Widom distribution functions.

Denote $\hat{\mathcal{A}}(t) = \mathcal{A}(t) - t^2$.

We first review two known identities. The first one is proved by Johansson in [52]:

$$(5.1) \qquad \mathbb{P}\left(2^{2/3} \sup_{t \in \mathbb{R}} \hat{\mathcal{A}}(t) \leq x \right) = F_{GOE}(x)$$

for every $x \in \mathbb{R}$. The second one is proved by Quastel and Remenik in [67]

$$(5.2) \qquad \mathbb{P}\left(\sup_{t \leq w} \hat{\mathcal{A}}(t) \leq x - \min\{0, w\}^2\right) = G_w^{2\to1}(x)$$

for every $w \in \mathbb{R}$ and $x \in \mathbb{R}$, where $G_w^{2\to1}(x)$ is the marginal distribution function of the process $\mathcal{A}_{2\to1}$ introduced in [22] (see (1.7) of [67]).

In this chapter, we present the following five new identities. If $\mathcal{B}(t)$ is a Brownian motion, denote

$$(5.3) \qquad \hat{\mathcal{B}}(t; w) = \mathcal{B}(t) - 2\sqrt{2}wt.$$

Theorem V.1. *(a) ([14]) Let $\mathcal{A}^{(1)}(t)$ and $\mathcal{A}^{(2)}(t)$ be two independent Airy processes. Then*

$$(5.4) \qquad \mathbb{P}\left(\sup_{t \in \mathbb{R}}\left(\frac{\alpha^{1/3}\hat{\mathcal{A}}^{(1)}(\alpha^{-2/3}t) + \beta^{1/3}\hat{\mathcal{A}}^{(2)}(\beta^{-2/3}t)}{(\alpha + \beta)^{1/3}}\right) \leq x\right) = F_{GUE}(x)$$

for every $\alpha, \beta > 0$ and for every $x \in \mathbb{R}$.

(b) ([15]) Let $\mathcal{A}(t)$ be the Airy process and $\mathcal{A}(-t)$ be its time reversal, then

$$(5.5) \qquad \mathbb{P}\left(\sup_{t \in \mathbb{R}}\left(\frac{\hat{\mathcal{A}}(t) + \hat{\mathcal{A}}(-t)}{2}\right) \leq x\right) = F_{GUE}(x)^2$$

for every $x \in \mathbb{R}$.

(c) ([30, 15]) Let $\mathcal{B}_1(t), \mathcal{B}_2(t), \cdots, t \geq 0$ be independent standard Brownian motions, then

$$(5.6) \qquad \mathbb{P}\left(\sup_{0=t_0 \leq t_1 \leq \cdots \leq t_k}\left(\hat{\mathcal{A}}(t_k) + \sqrt{2}\sum_{i=1}^{k}\hat{\mathcal{B}}_i(t_i; w_i) - \hat{\mathcal{B}}_i(t_{i-1}; w_i)\right) \leq x\right)$$
$$= F_k^{spiked}(x; w_1, \ldots, w_n)$$

for every $w_1, \ldots, w_k \in \mathbb{R}$ and every $x \in \mathbb{R}$, where $F_k^{spiked}(x; w_1, \ldots, w_n)$ is the distribution introduced in [7, Formula (54)] and [6, Corollary 1.3].

(d) ([30, 15]) Let $\mathcal{A}^{(1)}(t), \mathcal{A}^{(2)}(t)$ be two independent Airy processes, $\mathcal{B}_1(t), \mathcal{B}_2(t), \cdots, t \geq 0$ be independent two-sided Brownian motions with $\mathcal{B}_1(0) = \mathcal{B}_2(0) = \cdots = 0$, then

(5.7)

$$\mathbb{P}\left(\sup_{t_0 \leq t_1 \leq \cdots \leq ts_k}(\alpha^{\frac{1}{3}}\hat{\mathcal{A}}^{(1)}(\alpha^{-\frac{2}{3}}t_0) + \sqrt{2}\sum_{i=1}^{k}\hat{\mathcal{B}}_i(t_i; w_i) - \hat{\mathcal{B}}_i(s_{i-1}; w_i) + \beta^{\frac{1}{3}}\hat{\mathcal{A}}^{(2)}(\beta^{-\frac{2}{3}}s_k) \leq x\right)$$

$$= F_k^{spiked}(x; w_1, \ldots, w_n)$$

for every $\alpha, \beta > 0$, $w_1, \ldots, w_k \in \mathbb{R}$ and every $x \in \mathbb{R}$.

(e) ([30, 15]) Let $\mathcal{A}(t)$ be the Airy process and $\mathcal{B}(t)$ be the two-sided Brownian motion with $\mathcal{B}(0) = 0$, then

$$(5.8) \quad \mathbb{P}\left(\sup_{t \in \mathbb{R}}(\hat{\mathcal{A}}(t) + \sqrt{2}\mathcal{B}(t) + 4(w_+\mathbb{1}_{t<0} - w_-\mathbb{1}_{t>0})t) \leq x\right) \quad = \quad F_{st}(x; w_+, w_-)$$

for every $w_+, w_- \in \mathbb{R}$ and every $x \in \mathbb{R}$, where $F_{st}(x; w_+, w_-)$ is the distribution function introduced in [16].

There are at least three different ways to understand these identities.

First of all, it is conjectured that there is a relation between the Airy process and KPZ equation. More precisely, the Airy process is the limit of the solution to certain specific stochastic heat equation, which arises from the continuum random polymer. By using this relation, one can naturally obtain (5.1), (5.2) and (5.8). See [66] for details. It will be quite interesting to understand other identities in the KPZ language.

Second, as we mentioned in the introduction chapter, the Airy process is the limiting process of the top curve of certain non-intersecting processes. Therefore (5.1) means that the height of non-intersecting Brownian excursions converges to F_{GOE} [77]. Similarly (5.4) implies the width of non-intersecting Brownian bridges converges

F_{GUE} [14]. It is still open how to obtain other identities from the non-intersecting models.

Finally, all these identities can be understood in solvable directed last passage percolation (DLPP) models. DLPP models are defined as follows: Consider the lattice sites $\mathbb{Z}_+^2$. Suppose each site is associated with a random weight $w(i,j)$ for all $i,j \geq 1$. Denote by $\Pi_{(M,N)}$ the set of up/right paths from $(1,1)$ to (M,N) and define

$$(5.9) \qquad G(M,N) := \max_{\pi \in \Pi_{(M,N)}} \sum_{(i,j) \in \pi} w(i,j),$$

which is called the (directed) point to point last passage time.

The DLPP model is a specific 2-d random growth model, which can be thought as a randomly growing Young diagram [48]. When the entries are i.i.d. exponential variables, this DLPP model is equivalent to the totally asymmetric exclusion process [50]. There has been huge progress on the asymptotic behavior of the DLPP model. The first breakthrough is [9] where the author considered the longest increasing subsequence which is equivalent to the so-called Poisson DLPP model [5]. Then [50] considered the DLPP model when all the entries are i.i.d. geometric variables or i.i.d. exponential variables. In all the cases above, the limiting fluctuation of the point to point last passage percolation is Tracy-Widom distribution F_{GUE}.

Recall that F_{GUE} is the marginal distribution of the Airy process. In the DLPP model with i.i.d. geometric or exponential entries, it is proved that the process defined by the (rescaled) last passage time from one fixed point to multiple points converges to the Airy process. More precisely, the process $G(N+t, N-t), |t| < N$ after rescaling converges to the process $\hat{\mathcal{A}}$, the Airy process minus a parabola, as $N \to \infty$. This convergence is not only in the finite dimensional sense but also in the functional limit sense [52].

Besides the cases when all the entries are i.i.d. as we mentioned above, there are also many results when the entries are not i.i.d. variables. For examples, the asymptotics of $G(M, N)$ was obtained in [17] when the entries contain certain reflective/rotational symmetry. [16] considered the Poisson DLPP with heavy weights on the left and bottom edges, which is equivalent to the DLPP with i.i.d. geometric variables except that the entries on the two edges are geometric variables of different weights. [7] considered the DLPP model with i.i.d. exponential variables except that the entries on the finitely many rows at the bottom are exponential variables of different parameters. This result is also valid for the geometric variable case [47].

Now we explain how one can obtain the identities (5.1), (5.2), (5.4)-(5.8) in the solvable DLPP models. The identity (5.2) can be obtained by considering a point-to-half line last passage time by using the result of [22]. (5.4) arises naturally if one considers the point to point directed last passage time with i.i.d. geometric entries in two different methods. Similar considerations in the DLPP model with i.i.d. geometric entries with reflectional/rotational symmetry give rise to (5.1) and (5.4). The identities (5.6) and (5.7) follow from the DLPP model with i.i.d. geometric entries in each row where several finite rows are of special parameters. Similarly (5.8) follows from the DLPP model with i.i.d. geometric entries except for the first row and the first column.

The above explanation gives an indirect method to prove these identities. This idea was first used to prove the identity (5.1) by Johansson [52]. In this dissertation, we will modify Johansson's approach and prove the identities in Theorem V.1 [14, 15, 30].

The proofs of (5.4) and (5.5) both use a functional limit result that the point to the diagonal line last passage time in the DLPP model with i.i.d. geometric entries

converges to $\hat{\mathcal{A}}$. And the proofs (5.6), (5.7) and (5.8) use a similar functional limit result for the point to horizontal/vertical line last passage time in the same DLPP model, which we prove in section 5.3. We only present the proofs of (5.4) and (5.6), other identities in Theorem V.1 can be obtained similarly.

Remark V.2. A direct proof of (5.1) is given in [31] by using a new Fredholm determinant formula of $\mathbb{P}(\mathcal{A}(t) \leq g(t), t \in [-T, T])$ for a general function g and constant $T > 0$. This method also works for (5.2). It is interesting to find direct proofs of the identities in Theorem V.1.

Remark V.3. In [30], the authors prove a so-called uniform slow decorrelation property for the DLPP model with i.i.d. geometric entries. (5.6), (5.7), and (5.8) can be obtained by using this property. However, in this dissertation, we only prove a weak version of the property which is sufficient for the proof of these identities.

5.2 Proof of (5.4)

By symmetry we may assume $\alpha \leq \beta$. Let $w(i, j), (i, j) \in \mathbb{Z}_+^2$, be independent random variables with geometric distribution with parameter $1 - q$, i.e., $\mathbb{P}(w(i, j) = k) = (1 - q)q^k$, $k = 0, 1, 2, \cdots$. The limiting fluctuations of $G(M, N)$ are known to be F_{GUE} in [50] as M and N tend to infinite with a finite ratio. In particular, when $M = N = (\alpha + \beta)n$,

$$(5.10) \qquad \lim_{n \to \infty} \mathbb{P}\left(\frac{G((\alpha + \beta)n, (\alpha + \beta)n) - \mu(\alpha + \beta)n}{\sigma(\alpha + \beta)^{1/3}n^{1/3}} \leq x\right) = F_{GUE}(x),$$

where

$$(5.11) \qquad \mu = \frac{2\sqrt{q}}{1 - \sqrt{q}}, \qquad \sigma = \frac{q^{1/6}(1 + \sqrt{q})^{1/3}}{1 - \sqrt{q}}.$$

Consider the lattice points on the line connecting the points $(1, 2\alpha n)$ and $(2\alpha n, 1)$, i.e. $\mathcal{L} := \{(\alpha n + u, \alpha n - u) : |u| < \alpha n\}$. An up/right path from $(1, 1)$ to $((\alpha + \beta)n, (\alpha +$

$\beta)n)$ passes through a point on $\mathcal{L}$. Considering the up/right path from $(1,1)$ to a point on $\mathcal{L}$ and the down/left path from $((\alpha+\beta)n, (\alpha+\beta)n)$ to the same point on $\mathcal{L}$ (see Figure 5.1), we find that $G((\alpha+\beta)n, (\alpha+\beta)n)$ equals

$$(5.12) \qquad \max_{|u|<\alpha n} \left(G^{(1)}(\alpha n+u, \alpha n-u) + G^{(2)}(\beta n+u, \beta n-u) \right) + Err,$$

where $G^{(1)}$ and $G^{(2)}$ are two independent copies of G, and the error term Err comes from the duplicate diagonal term $w(\alpha n+u, \alpha n-u)$.
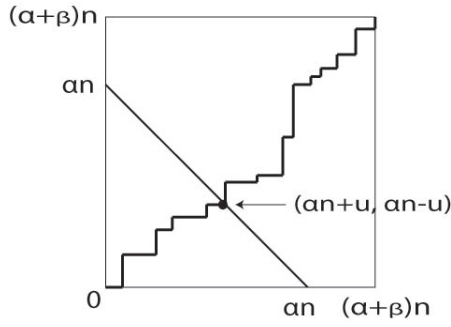

Figure 5.1: Intersection of an up/right path with $\mathcal{L}$

Consider $G^{(i)}(\alpha n+u, \alpha n-u)$ as a process in time u. For u of order $n^{2/3}$, it was shown in [52] that the fluctuations of this process converge the Airy process in the functional convergence. More precisely, if we set

$$(5.13) \qquad H_n^{(1)}(t) := \frac{G^{(1)}(\alpha n+d^{-1}(\alpha n)^{2/3}t, \alpha n-d^{-1}(\alpha n)^{2/3}t) - \mu\alpha n}{\sigma(\alpha n)^{1/3}},$$

and

$$(5.14) \qquad H_n^{(2)}(t) := \frac{G^{(2)}(\beta n+d^{-1}(\beta n)^{2/3}t, \beta n-d^{-1}(\beta n)^{2/3}t) - \mu\beta n}{\sigma(\beta n)^{1/3}},$$

for $|t| < d(\alpha n)^{1/3}$, where $d := q^{1/6}(1+\sqrt{q})^{-2/3}$, then $H_n^{(i)}(t)$ converges to the Airy process $\hat{\mathcal{A}}^{(i)}(t) = \mathcal{A}^{(i)}(t) - t^2$, $i=1,2$. (We note that there is a typographical error in the formula (1.8) in [52] where, in terms of our notations, σ is changed to $\frac{q^{1/6}(1+\sqrt{q})^{1/3}}{1-q}$. However, the correct formula of σ is $\frac{q^{1/6}(1+\sqrt{q})^{1/3}}{1-\sqrt{q}}$ as in (5.11) which is also same as

in [50].) Note that the error term in (5.12) $Err = O((\log n)^{1+\epsilon})$ with probability 1 as N tends to infinity for any $\epsilon > 0$, therefore

$$
\begin{aligned}
(5.15) \quad & \lim_{N\to\infty} \mathbb{P}\left(\frac{G(N,N) - \mu N}{\sigma N^{1/3}} \leq x\right) \\
& = \lim_{N\to\infty} \mathbb{P}\left(\max_{|t|<d\alpha n^{1/3}} \left(\alpha^{1/3} H_n^{(1)}(\alpha^{-2/3}t) + \beta^{1/3} H_n^{(2)}(\beta^{-2/3}t)\right) \leq (\alpha+\beta)^{1/3}x\right).
\end{aligned}
$$

We obtain (5.4) if we prove that

$$
\begin{aligned}
(5.16) \quad & \lim_{n\to\infty} \mathbb{P}\left(\max_{|t|<d\alpha n^{1/3}} \left(\alpha^{1/3} H_n^{(1)}(\alpha^{-2/3}t) + \beta^{1/3} H_n^{(2)}(\beta^{-2/3}t)\right) \leq (\alpha+\beta)^{1/3}x\right) \\
& = \mathbb{P}\left(\max_{t\in\mathbb{R}} \left(\alpha^{1/3} \hat{\mathcal{A}}^{(1)}(\alpha^{-2/3}t) + \beta^{1/3} \hat{\mathcal{A}}^{(2)}(\beta^{-2/3}t)\right) \leq (\alpha+\beta)^{1/3}x\right).
\end{aligned}
$$

In [52], a similar identity

$$
(5.17) \quad \lim_{n\to\infty} \mathbb{P}\left(\max_{|t|<dn^{1/3}} H_n(t) \leq x\right) = \mathbb{P}\left(\max_{t\in\mathbb{R}} \hat{\mathcal{A}}(t) \leq x\right)
$$

was proved as a part of the proof of (5.1). We proceed similarly and use the estimates obtained in [52] .

Set

$$
(5.18) \quad X_{n,T} := (\alpha+\beta)^{-1/3} \cdot \max_{|t|\leq T} \left(\alpha^{1/3} H_n^{(1)}(\alpha^{-2/3}t) + \beta^{1/3} H_n^{(2)}(\beta^{-2/3}t)\right)
$$

and

$$
(5.19) \quad Y_{n,T} := (\alpha+\beta)^{-1/3} \cdot \max_{|t|>T} \left(\alpha^{1/3} H_n^{(1)}(\alpha^{-2/3}t) + \beta^{1/3} H_n^{(2)}(\beta^{-2/3}t)\right).
$$

Since

$$
\begin{aligned}
(5.20) \quad & \mathbb{P}(X_{n,T} \leq x) \\
& \geq \mathbb{P}\left(\max_{|t|<d\alpha n^{1/3}} \left(\alpha^{1/3} H^{(1)}(\alpha^{-2/3}t) + \beta^{1/3} H_n^{(2)}(\beta^{-2/3}t)\right) \leq (\alpha+\beta)^{1/3}x\right) \\
& \geq \mathbb{P}(X_{n,T} \leq x) - \mathbb{P}(Y_{n,T} > x)
\end{aligned}
$$

for all large enough n for each fixed T, (5.16) follows from the following three properties:

(a) For each $\epsilon > 0$, there are positive constants T_0 and n_0 such that $\mathbb{P}\left(Y_{n,T} > x\right) < \epsilon$

for all $T > T_0$ and $n > n_0$,

(b) For each fixed T, $\mathbb{P}\left(X_{n,T} \leq x\right) \to \mathbb{P}\left(A_T \leq x\right)$ as $n \to \infty$.

(c) Finally, $\mathbb{P}(A_T \leq s) \to \mathbb{P}(A_\infty \leq x)$ as $T \to \infty$.

Here

$$(5.21) \qquad A_T := (\alpha + \beta)^{-1/3} \cdot \max_{|t| \leq T} \left(\alpha^{1/3} \hat{\mathcal{A}}^{(1)}(\alpha^{-2/3}t) + \beta^{1/3} \hat{\mathcal{A}}^{(2)}(\beta^{-2/3}t) \right)$$

and A_∞ is the same random variable with the maximum taken over $t \in \mathbb{R}$.

A functional limit theorem to the Airy process was proved in [52] (Theorem 1.2). This means that $H_n^{(i)}(t) \to \hat{\mathcal{A}}^{(i)}(t)$ at $n \to \infty$ in the sense of weak convergence of the probability measures on $C[-T, T]$ for each fixed T. Hence the property (b) follows a theorem on the convergence of product measures ([21], Theorem 3.2).

The property (c) follows from the monotone convergence theorem since $\{A_\infty \leq s\} = \cap_{T>0}\{A_T \leq s\}$.

For the property (a), we use the estimates (5.19) and (5.20) in [52]: there are positive constants C and c such that

$$(5.22) \quad \begin{aligned} &\mathbb{P}\left(\max_{T < t \leq \log n} H_n^{(i)}(\alpha^{-2/3}t) > M \right) \\ &\leq \int_{\alpha^{-2/3}T - 1}^\infty e^{-c(M-1+x^2)^{3/2}}\,dx + C \int_{\alpha^{-2/3}T - 1}^\infty e^{-x^3}\,dx \end{aligned}$$

and

$$(5.23) \qquad \mathbb{P}\left(\max_{t \geq \log n} H_n^{(i)}(\alpha^{-2/3}t) > M \right) \leq Cne^{-c(\log n)^3}$$

for all M. Therefore, taking $M = \alpha^{-1/3}(\alpha + \beta)^{1/3}s/2$, for any $\epsilon > 0$, we have

$$(5.24) \qquad \mathbb{P}\left((\alpha + \beta)^{-1/3} \max_{t \geq T} \alpha^{1/3} H_n^{(i)}(\alpha^{-2/3}t) > \frac{s}{2} \right) < \frac{\epsilon}{2},$$

if T, n are both large enough. This proves (a).

5.3 Proof of (5.6)

5.3.1 Two Lemmas on the DLPP Model with i.i.d. Geometric Random Variables

Consider the following DLPP model: each site $(i,j), i, j \in \mathbb{Z}_+$ is associated with an i.i.d. random geometric variable with parameter $1 - q$, i.e., $\mathbb{P}(w(i,j) = k) = (1-q)q^k$ for $k = 0, 1, \cdots$. Define

$$(5.25) \qquad H_N(t) := \frac{G(N + d^{-1}N^{2/3}t, N - d^{-1}N^{2/3}t) - \mu N}{\sigma N^{1/3}}$$

for all $|t| < dN^{1/3}$, and

$$(5.26) \qquad \tilde{H}_N(t) := \frac{G(N - 2d^{-1}N^{2/3}t, N) - \mu(N - d^{-1}N^{2/3}t)}{\sigma N^{1/3}}$$

for all $t < \frac{dN^{1/3}}{2}$. Here $d = q^{1/6}(1 + \sqrt{q})^{-2/3}$, and μ, σ are defined in (5.11). It is well known that

$$(5.27) \qquad \qquad H_N(t) \to \hat{\mathcal{A}}(t)$$

in any fixed interval $[-T, T]$ in the sense of weak star topology on $C[-T, T]$.

We need the following two lemmas about H_N and $\tilde{H}_N$:

Lemma V.4. *Suppose $0 < \lambda < 1$ be any fixed constant. For any $\epsilon > 0, x \in \mathbb{R}$, there exist some constant $N_0 := N_0(\epsilon, \lambda, x)$ and $M_0 := M_0(\epsilon, \lambda, x)$ such that*

$$(5.28) \qquad \mathbb{P}\left(\max_{|t| > M}(H_N(t) + \lambda t^2) > x \right) < \epsilon$$

for arbitrary $N \geq N_0, M \geq M_0$.

Lemma V.5. *For any fixed T, we have*

$$(5.29) \qquad \qquad \lim_{N \to \infty} \tilde{H}_N(t) \to \hat{\mathcal{A}}(t)$$

in $[-T, T]$, in sense of weak star topology on $C[-T, T]$.

Proof of Lemma V.4. The case when $\lambda = 0$ is proven in [52]. And it is possible to modify the proof for the case $\lambda \in (0, 1)$. However, we use a different technique to prove this lemma. This specific DLPP model has a called Gibbs property of point to diagonal line last passage time, which is proven in [30] by using the techniques developed by [29]. It claims that if we have three collinear points (K_1, M_1), (K_2, M_2) and (K_3, M_3), $(K_1 < K_2 < K_3)$, then

(5.30)
$$\mathbb{P}\left(\max_{j\in[K_1,\frac{1}{2}(K_1+K_2)]} G(N-j, N+j) \geq M_1\right)$$
$$\leq 2\mathbb{P}\left(G(N+K_2, N-K_2) \geq M_2\right) + \mathbb{P}\left(G(N+K_3, N-K_3) \leq M_3\right),$$

where $c \in (0, 1)$ is any fixed constant. We will use this property to prove Lemma V.4.

Consider the following collinear points

(5.31)
$$(d^{-1}N^{2/3}t, \mu N + \sigma N^{1/3}(x - \lambda(t+1)^2)),$$
$$(d^{-1}N^{2/3}(t+2), \mu N + \sigma N^{1/3}(x - \frac{\lambda+2}{3}(t+1)^2)),$$
$$(d^{-1}N^{2/3}(t+6), \mu N + \sigma N^{1/3}(x - (2-\lambda)(t+1)^2)).$$

By using (5.30), we have

(5.32)
$$\mathbb{P}\left(\max_{d^{-1}N^{2/3}t\leq s\leq d^{-1}N^{2/3}(t+1)} G(N+s, N-s) \geq \mu N + \sigma N^{1/3}(x - \lambda(t+1)^2)\right)$$
$$\leq 2\mathbb{P}\left(G(N+d^{-1}N^{2/3}(t+2), N-d^{-1}N^{2/3}(t+2)) \geq \mu N + \sigma N^{1/3}(x - \frac{\lambda+2}{3}(t+1)^2))\right)$$
$$+ \mathbb{P}\left(G(N+d^{-1}N^{2/3}(t+6), N-d^{-1}N^{2/3}(t+6)) \geq \mu N + \sigma N^{1/3}(x - (2-\lambda)(t+1)^2)\right).$$

The following tail estimate of $H_N(s)$ is known (Claim 5.7 [52])

(5.33)
$$\mathbb{P}\left(H_N(s) \geq y\right) \leq e^{-c(y+s^2)^{3/2}}$$

for all $s^2 + y \geq M_1$ and $N \geq N_1$, where M_1, N_1 are both large enough and $c :=$

$c(M_1, N_1)$ is a positive constant which only depends on M_1, N_1. Therefore

(5.34)

$$\mathbb{P}\left(G(N + d^{-1}N^{2/3}(t+2), N - d^{-1}N^{2/3}(t+2)) \geq \mu N + \sigma N^{1/3}(x - \frac{\lambda+2}{3}(t+1)^2))\right)$$

$$\leq e^{-c(x + \frac{1-\lambda}{3}(t+2)^2)^{3/2}}.$$

Suppose $t \leq \log N$. By Theorem 1.1 [10] we have when N is large enough ($N \geq N_2$ for example)

$$(5.35) \qquad\qquad \mathbb{P}\left(H_N(t+6) \leq -y - (t+6)^2\right) \leq e^{-c'y^3}$$

uniformly for $y \in [L, \delta N^{2/3}]$ and $t \leq \log N$, where L is a constant independent of N, and c' is a positive constant independent of t. Therefore

$$(5.36) \qquad \mathbb{P}\left(H_N(t+6) \leq x - (2-\lambda)(t+1)^2\right) \leq e^{-c'((2-\lambda)(t+1)^2 - (t+6)^2 - x)^3}$$

if $(2-\lambda)(t+1)^2 - (t+6)^2 - x > L$. Note $\lambda < 1$, $(2-\lambda)(t+1)^2 - (t+6)^2 - x > L$ always holds if $t \geq M_1 \geq M_2 = M_2(\lambda, x)$.

By plugging (5.34) and (5.36) into (5.32) we have

(5.37)

$$\mathbb{P}\left(\max_{t \leq s \leq t+1} H_N(s) + \lambda(t+1)^2 \geq x\right) \leq 2e^{-c(x + \frac{1-\lambda}{3}(t+2)^2)^{3/2}} + e^{-c'((2-\lambda)(t+1)^2 - (t+6)^2 - x)^3}$$

for arbitrary $N \geq \max\{N_1, N_2\}$ and $M_1 \leq t \leq \log N$. Hence

(5.38)

$$\mathbb{P}\left(\max_{M_1 \leq t \leq \log N}(H_N(s) + \lambda t^2) \geq x\right) \leq \sum_{t=M_1}^{\infty}\left(2e^{-c(x + \frac{1-\lambda}{3}(t+2)^2)^{3/2}} + e^{-c'((2-\lambda)(t+1)^2 - (t+6)^2 - x)^3}\right)$$

for all $N \geq \max\{N_1, N_2\}$ and $M_1 \geq M_2(\lambda, x)$. Since the above sum is bounded, there exists $M_3 = M_3(\epsilon, \lambda, x)$ such that

$$(5.39) \qquad\qquad \mathbb{P}\left(\max_{M \leq t \leq \log N}(H_N(t) + \lambda t^2) \geq x\right) \leq \frac{\epsilon}{2}$$

for all $N \geq \max\{N_1, N_2\}$ and $M \geq M_3$.

Now we apply (5.33) again,

$$(5.40) \qquad \mathbb{P}\left(\max_{t \geq \log N}(H_N(t) + \lambda t^2) \geq x\right) \leq Ne^{-c(x+(1-\lambda)(\log N)^2)^{3/2}} \leq \frac{\epsilon}{2}$$

when $N \geq N_3 = N_3(\epsilon, \lambda, x)$.

(5.39) and (5.40) imply that

$$(5.41) \qquad\qquad\qquad \mathbb{P}\left(\max_{t \geq M}(H_N(t) + \lambda t^2) \geq x\right) \leq \epsilon$$

for all $M \geq M_4(\epsilon, \lambda, x)$ and $N \geq N_4(\epsilon, \lambda, x)$.

By symmetry Lemma V.4 follows.

$\square$

Proof of Lemma V.5. In [30], the authors proved the so-called uniform slow decorrelation, which asserts that $\tilde{H}_N(t) - H_N(t) \to 0$ uniformly in $[-T, T]$ as N tends to infinity. Therefore Lemma V.5 follows immediately from the uniform slow decorrelation and (5.27).

For this specific case, we give an independent proof. This proof can be modified to prove the uniform slow decorrelation mentioned above, see [30] for more details. Define

(5.42)
$$H_{(N,-)}(t) := \frac{G(N - d^{-1}N^{2/3}(2T+t), N - d^{-1}N^{2/3}(2T-t)) - \mu(N - 2d^{-1}N^{2/3}T)}{\sigma N^{1/3}},$$

and

(5.43)
$$H_{(N,+)}(t) := \frac{G(N + d^{-1}N^{2/3}(2T-t), N + d^{-1}N^{2/3}(2T+t)) - \mu(N + 2d^{-1}N^{2/3}T)}{\sigma N^{1/3}}$$

for all $t \in [-T, T]$. By (5.27), both processes $H_{(N,\pm)}$ are tight. Together with the slow decorrelation property [28] we have the following: for any given $\epsilon, \gamma > 0$, there

exist $\delta > 0$ and an integer N_0 such that

(5.44) $$\mathbb{P}\left(\max_{|t|,|s|\leq T,|s-t|\leq\delta}|H_{(N,\pm)}(t)-H_{(N,\pm)}(s)| \geq \gamma\right) \leq \epsilon$$

hold for all $N > N_0$. Here the two indices $\pm$ are not necessary the same.

Now we compare $\tilde{H}_N$ with $H_{(N,\pm)}$. Note that

$$G(N-2d^{-1}N^{2/3}t, N) \geq G(N-d^{-1}N^{2/3}(2T+t), N-d^{-1}N^{2/3}(2T-t))$$
(5.45)
$$+ G_{(N-d^{-1}N^{2/3}(2T+t),N-d^{-1}N^{2/3}(2T-t))}(N-2d^{-1}N^{2/3}t, N),$$

where $G_{(i,j)}(i',j')$ denotes the point to point last passage time from the site (i,j) to the site (i',j'). Together with (5.35) we immediately obtain

(5.46) $$\mathbb{P}\left(\tilde{H}_N(t)-H_{(N,-)}(t) \leq -\gamma\right) \leq e^{-c\gamma^3 N^{1/3}}$$

uniformly for $t \in [-T,T]$ and N large enough.

Similarly we have

(5.47) $$\mathbb{P}\left(\tilde{H}_N(t)-H_{(N,+)}(t) \geq \gamma\right) \leq e^{-c\gamma^3 N^{1/3}}$$

uniformly for $t \in [-T,T]$ and N large enough. These two inequalities and (5.44) imply the tightness of $\tilde{H}_N(t)$. Note that (5.29) holds in the sense of finite dimensional distribution [28]. Therefore it holds in the sense of functional limit. □

5.3.2   The Proof of (5.6) by Using Lemmas V.4 and V.5

Now we prove (5.6). We only prove the case when $k = 1$. The case when $k > 1$ is similar.

Consider the point to point last passage time from $(1,1)$ to $(N, N+1)$ in the DLPP model with entries $w(i,j)$, which are independent geometric random variables with parameter $1 - \sqrt{q}(1 - 2w\sigma^{-1}N^{-1/3})$ (if $j = N + 1$) and $1 - q$ (if $j = 1, 2, \cdots, N$).

This model has been considered in [6, 47] and the limiting fluctuation of $G(N, N+1)$ is given by

$$(5.48) \qquad \lim_{N\to\infty} \mathbb{P}\left(\frac{G(N, N+1) - \mu N}{\sigma N^{1/3}} \le x\right) = F_{st}^{spiked}(x; w).$$

Now we evaluate $G(N, N+1)$ in a different way. Note that any directed path from $(1, 1)$ to $(N, N+1)$ will intersect the line $\{(i, N)|i = 1, 2, \cdots, N\}$. Then $G(N, N+1)$ can be written as

$$(5.49) \qquad G(N, N+1) = \max_{i=1,2,\cdots,N} (G(i, N) + S_{N+1-i})$$

where $S_{N+1-i} := w(i, N+1) + \cdots + w(N, N+1)$ is the sum of $(N+1-i)$ i.i.d. random geometric variables. Therefore

$$(5.50) \qquad \mathbb{P}\left(\frac{G(N, N+1) - \mu N}{\sigma N^{1/3}} \le x\right) = \mathbb{P}\left(\max_{t\in[0,\frac{dN^{1/3}}{2}]} \tilde{H}_N(t) + \mathcal{P}_N(t) \le x\right),$$

where

$$(5.51) \qquad \mathcal{P}_N(t) := \frac{S_{1+2d^{-1}tN^{2/3}} - \mu d^{-1} N^{2/3}}{\sigma N^{1/3}}.$$

Note that by Donsker's Theorem, we have

$$(5.52) \qquad \mathcal{P}_N(t) \to \sqrt{2}\hat{\mathcal{B}}(t)$$

in any fixed interval $[0, T]$ in the sense of weak star topology on $C[0, T]$, as N tends to infinity.

Similarly to the proof of (5.4), it is easy to show (5.6) follows from Lemma V.5, (5.52) and the following claim.

Claim V.6. *(a) For any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exist positive constants T_0 and N_0 such that*

$$(5.53) \qquad \mathbb{P}\left(\max_{t\ge T} \left(\tilde{H}_N(t) + \lambda t^2\right) > \frac{x}{2}\right) < \epsilon$$

for all $T > T_0$ and $N > N_0$.

(b) For any $\epsilon > 0$, there exist positive constants T_1 and N_1 such that

$$(5.54) \qquad \mathbb{P}\left(\max_{t \geq T}\left(\mathcal{P}_N(t) - \lambda t^2\right) > \frac{x}{2}\right) < \epsilon$$

for all $T > T_1$ and $N > N_1$.

The rest is to prove the Claim V.6. (b) is easy by the property of shifted Brownian motion, so we leave it as an exercise. For part (a), the key idea is to compare $\tilde{H}_N(t)$ with $H_N(t)$, where $H_N(t)$ is defined in (5.25) in the DLPP model with random i.i.d. geometric entries such that the entry at $(i,j), 1 \leq i, j \leq N$, is exact $w(i,j)$.

Since

$$(5.55) \qquad \begin{aligned} &G(N - d^{-1}N^{2/3}t, N + d^{-1}N^{2/3}t) \\ &\geq G(N - 2d^{-1}N^{2/3}t, N) + \tilde{G}_{(N-2d^{-1}N^{2/3}t,N)}(N - d^{-1}N^{2/3}t, N + d^{-1}N^{2/3}t), \end{aligned}$$

where $\tilde{G}_{(i,j)}(i', j')$ denotes the point to point last passage time from the site (i, j) to the site (i', j') in the DLPP model with random i.i.d. geometric entries with parameter $1 - q$, we immediately have

$$(5.56)$$
$$\mathbb{P}\left(\max_{t \geq T}(\tilde{H}_N(s) + \lambda t^2) > \frac{x}{2}\right) \leq \mathbb{P}\left(\max_{t \geq T}(H_N(t) + \lambda t^2) > \frac{x}{2} - 1\right)$$
$$+ \mathbb{P}\left(\min_{T \leq t \leq \frac{dN^{1/3}}{2}}\left(\tilde{G}_{(N-2d^{-1}N^{2/3}t,N)}(N - d^{-1}N^{2/3}t, N + d^{-1}N^{2/3}t) - \mu d N^{2/3}t\right) \leq -\sigma N^{\frac{1}{3}}\right).$$

On the other hand, the lower tail estimate (5.35) implies

$$(5.57)$$
$$\mathbb{P}\left(\min_{T \leq t \leq \frac{dN^{1/3}}{2}}\left(\tilde{G}_{(N-2d^{-1}N^{2/3}t,N)}(N - d^{-1}N^{2/3}t, N + d^{-1}N^{2/3}t) - \mu d N^{2/3}t\right) \leq -\sigma N^{\frac{1}{3}}\right)$$
$$\leq N c' e^{-N^{c'}}$$

for large enough N, where c, c' are both positive constant independent of N. Combining the above estimate and Lemma V.4, we obtain (5.53).

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